FIN 539 MATHEMATICAL FINANCE
Lecture 3: Multiple assets, hedging state variables, and pricing models

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Mini math review: multidimensional Itô's lemma

Let $H : \Re^d \times [0,T] \to \Re$ with continuous partial derivatives $H_x(x,t)$, $H_{xx}(x,t)$, and $H_t(x,t)$. Let $dX_t = g(t)dt + G(t)dZ_t$, where X_t is a d-dimensional process and Z_t is an m-dimentional standard Wiener process. Then $Y_t \equiv H(X_t,t)$ is an Itô process with stochastic differential

$$dY_t = H_t dt + H_x' dX + \frac{1}{2}\mathrm{tr}(GG'H_{xx})dt$$

where, for any symmetric matrix A, $\mathrm{tr}(A)$ denotes the trace, which is the sum $\Sigma_i A_{ii}$ of its diagonal elements.

Note: if H takes values in $\Re^k$, we can apply the result elementwise.

What is the trace?

The trace $\mathrm{tr}(A)$ of the square matrix A is the sum of its diagonal elements, $\Sigma_i A_{ii}$. The trace equals the sum of the eigenvalues.[1] For matrices A $i \times j$ and B $j \times i$, then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. For matrices C $i \times j$, D $j \times k$, and E $k \times i$, $\mathrm{tr}(CDE) = \mathrm{tr}(DEC) = \mathrm{tr}(ECD)$. If F and G both $n \times n$, $\mathrm{tr}(F + G) = \mathrm{tr}(F) + \mathrm{tr}(G)$ and $\mathrm{tr}(F') = \mathrm{tr}(F)$. Also, if X is $n \times n$, $d(\mathrm{tr}(X))/dX = I_{n\times n}$ and $d(\mathrm{tr}(AB))/dA = B'$.

[1] It is also useful to know that the determinant is the product of the eigenvalues.

Multidimensional portfolio example

Consider an infinite-horizon portfolio problem with wealth dynamics

$$dw = rwdt + \theta'((\mu - r\mathbf{1})dt + \Gamma dZ) - cdt$$

where θ is $n \times 1$, Γ is $n \times k$, and Z is $k \times 1$. Given

$$M_t = \int_{s=0}^{t} e^{-\rho s} u(c_s) ds + e^{-\rho t} V(w_t),$$

we can compute

$$E\left[\frac{dM}{e^{-\rho t}}\right] = \left(u(c) - \rho V + (rw + \theta'(\mu - r\mathbf{1}) - c)V_w + \frac{1}{2}\mathrm{tr}(\theta'\Gamma\Gamma'\theta V_{ww})\right) dt$$

$\theta'\Gamma\Gamma'\theta$ and V_{ww} are scalars, so the Bellman equation is

$$\max_{c,\theta}\left(u(c) - \rho V + (rw + \theta'(\mu - r\mathbf{1}) - c)V_w + \frac{1}{2}\theta'\Gamma\Gamma'\theta V_{ww}\right) = 0$$

Assume that $\Gamma\Gamma'$ is positive definite.

Multidimensional portfolio example: continued

Therefore, the terms involving θ are

$$\theta'(\mu - r\mathbf{1})V_w + \frac{1}{2}\theta'\Gamma\Gamma'\theta V_{ww},$$

where $\Gamma\Gamma'$ is the local covariance matrix of security returns. The first-order condition for optimal θ is

$$(\mu - r\mathbf{1})V_w + \Gamma\Gamma'\theta V_{ww} = 0$$

As before, $u'(c) = V_w$ so $c = I(V_w)$, but now the optimal portfolio is

$$\theta^* = -\frac{V_w}{V_{ww}}(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1})$$

The optimization is locally a mean-variance problem.

An example with more state variables

Given w_0 and S_0,
choose adapted θ_t, c_t, and w_t to
maximize $\mathrm{E}[\int_{t=0}^{\infty} e^{-\rho t} u(c_t)dt]$
s.t. $(\forall t)(dw_t = r(S_t)w_t dt + \theta_t'((\mu(S_t) - r(S_t)\mathbf{1})dt + \Gamma(S_t)dZ_t) - c_t dt + y(S_t)dt - L(S_t)dt)$
$(\forall t) w_t \geq 0$
$(\forall t) dS_t = a(S_t)dt + b(S_t)dZ_t$

In this problem, θ is $n \times 1$, Γ is $n \times k$, Z is $k \times 1$, S_t is $m \times 1$, a is $m \times 1$, and b is $m \times k$. However, c_t, w_t, y, and L are all scalar processes with units of wealth. The state variables are w_t and S_t, and we have

$$M_t = \int_{s=0}^{t} e^{-\rho s} u(c_s)ds + e^{-\rho t} V(w_t, S_t)$$

An example with more state variables: continued

$$
\begin{aligned}
\frac{\mathrm{E}[dM_t]}{e^{-\rho t}dt} &= u(c) - \rho V + V_w \frac{\mathrm{E}[dw]}{dt} + V_S' \frac{\mathrm{E}[dS]}{dt} \\
&\quad + \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} \theta'\Gamma \\ b \end{pmatrix} (\Gamma'\theta, b') \begin{pmatrix} V_{ww} & V_{wS} \\ V_{Sw} & V_{SS} \end{pmatrix} \right)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\mathrm{tr}\left(\begin{pmatrix} \theta'\Gamma \\ b \end{pmatrix} (\Gamma'\theta, b') \begin{pmatrix} V_{ww} & V_{wS} \\ V_{Sw} & V_{SS} \end{pmatrix} \right) = \mathrm{tr}\left(\begin{pmatrix} \theta'\Gamma\Gamma'\theta & \theta'\Gamma b' \\ b\Gamma'\theta & bb' \end{pmatrix} \begin{pmatrix} V_{ww} & V_{wS} \\ V_{Sw} & V_{SS} \end{pmatrix} \right) \\
&= \mathrm{tr} \begin{pmatrix} \theta'\Gamma\Gamma'\theta V_{ww} + \theta'\Gamma b' V_{Sw} & \theta'\Gamma\Gamma'\theta V_{wS} + \theta'\Gamma b' V_{SS} \\ b\Gamma'\theta V_{ww} + bb'V_{Sw} & b\Gamma'\theta V_{wS} + bb'V_{SS} \end{pmatrix} \\
&= \theta'\Gamma\Gamma'\theta V_{ww} + 2\theta'\Gamma b' V_{Sw} + \mathrm{tr}(bb'V_{SS}),
\end{aligned}
$$

(where we have used the fact that $\mathrm{tr}(b\Gamma'\theta V_{wS}) = \mathrm{tr}(V_{wS} b\Gamma'\theta) = \theta'\Gamma b' V_{Sw}$).
Also,

$$
E[dw]/dt = rw + \theta'(\mu - r\mathbf{1}) - c + y - L.
$$

An example with more state variables: continued 2

Therefore we have

$$\begin{aligned}\frac{\mathrm{E}[dM_t]}{e^{-\rho t}dt} &= u(c) - \rho V + (rw + \theta'(\mu - r\mathbf{1}) - c + y - L)V_w + V_S' a \\ &\quad + \frac{1}{2}(\theta'\Gamma\Gamma'\theta V_{ww} + 2\theta'\Gamma b' V_{Sw} + \mathrm{tr}(bb'V_{SS}))\end{aligned}$$

and the Bellman equation is

$$\begin{aligned}0 &= \max_{c,\theta}(u(c) - \rho V + (rw + \theta'(\mu - r\mathbf{1}) - c + y - L)V_w + V_S' a \\ &\quad + \frac{1}{2}(\theta'\Gamma\Gamma'\theta V_{ww} + 2\theta'\Gamma b' V_{Sw} + \mathrm{tr}(bb'V_{SS}))).\end{aligned}$$

As before, $u'(c) = V_w$ so $c = I(V_w)$, but now the optimal portfolio is

$$\theta^* = -\frac{V_w}{V_{ww}}(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}) - \frac{1}{V_{ww}}(\Gamma\Gamma')^{-1}\Gamma b' V_{Sw}$$

An example with more state variables: intepretation

$$\theta^* = -\frac{V_w}{V_{ww}}(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}) - \frac{1}{V_{ww}}(\Gamma\Gamma')^{-1}\Gamma b' V_{Sw}$$

The first term is the same as in a model without S and gives the optimal trade-off between risk and return. The second term is a hedging term. $(\Gamma\Gamma')^{-1}\Gamma b'$ is the regression coefficient vector of changes of the state variables on the asset returns. If these regression coefficients are all zero, then the asset returns cannot be used to hedge against the state variables. The other part, V_{Sw}/V_{ww}, tells us to what extent changes in wealth are substitutable for changes in the state variable. For example, if they are perfectly substitutable, $V(w,S) = v(w + \alpha' S)$ and we have that $V_{ww} = v''$ and $V_{Sw} = \alpha v''$ so $V_{Sw}/V_{ww} = \alpha$, and we would like to buy exposure of $-\alpha$ to S in the portfolio to undo the implicit exposure in the value function. However, if they are not substitutable at all and $V(w,S) = v_1(w) + v_2(S)$, then $V_{Sw}/V_{ww} = 0/{v_1}'' = 0$, and there is no point hedging because changes in wealth are not at all substitutable for changes in S and therefore no hedge is possible.

Asset pricing models

Some popular asset pricing models

- Capital Asset Pricing Model (CAPM): expected excess returns depend on covariance with market returns
- Consumption Capital Asset Pricing Model (CCAPM): expected excess returns depend on covariance with changes in the marginal utility of consumption
- Intertemporal Capital Asset Pricing Model (ICAPM): expected excess returns depend on covariance with market returns and covariance with factors describing changes in future investment opportunities
- Arbitrage Pricing Theory (APT): expected excess returns depend on covariance with common factors in stock returns

Roughly speaking, these are ordered from least general to most general.

Capital Asset Pricing Model (CAPM)

The CAPM is the most popular workhorse investment model in practice, and is probably the simplest to explain to people who are not quants. Even if you are using a more sophisticated model, I suggest linking your analysis to the CAPM for communications purposes. We think of having agents $q = 1, ..., Q$. From the model without state variables, we can write the first-order condition for optimal portfolio choice as

$$\frac{1}{-V_{ww}^q / V_w^q}(\mu - r\mathbf{1}) = \Gamma\Gamma'\theta^q.$$

Summing across agents q and dividing by Q, we have

$$\left(\frac{1}{Q}\sum_q \frac{1}{-V_{ww}^q / V_w^q}\right)(\mu - r\mathbf{1}) = \Gamma\Gamma'\left(\frac{1}{Q}\sum_q \theta^q\right),$$

where $(1/Q)\,\Sigma_q\,\theta^q$ is the per capita average risky portfolio.

Capital Asset Pricing Model (CAPM): continued

or

$$\mu - r\mathbf{1} = \left((1/Q)\sum_q (1/(-V^q_{ww}/V^q_w))\right)^{-} 1\Gamma\Gamma' \left(\frac{1}{Q}\sum_q \theta^q\right),$$

so expected excess return is proportional to covariance with the average portfolio times the harmonic mean absolute risk aversion. Writing the market portfolio of risky assets as $\theta^m \equiv \Sigma_q\, \theta^q / \Sigma_q\, \mathbf{1}'\theta^q$ and the vector of betas $\beta' \equiv (\theta^{m\prime}\Gamma\Gamma'\theta^m)^{-1}\theta^{m\prime}\Gamma\Gamma'$, the vector of excess returns is proportional to β, and because the market portfolio is weighted combination of the portfolios and the market's beta with respect to itself is 1, the constant of proportionality is the excess return on the market:

$$\mu - r\mathbf{1} = \beta(\theta^{m\prime}(\mu - r\mathbf{1})),$$

since the market's beta is $\beta'\theta^m = (\theta^{m\prime}\Gamma\Gamma'\theta^m)^{-1}\theta^{m\prime}\Gamma\Gamma'\theta^m = 1$.

Consumption Capital Asset Pricing Model (CCAPM)

The CCAPM says that excess returns equal covariance of returns with proportional changes in the marginal utility of consumption $u'(c) = V_w$. Applying Itô's lemma to $V_w(w_t, S_t)$ in the model with state variables and focusing on the random terms, we have that

$$\begin{aligned} du'(c) = dV_w^q(w_t, S_t) &= (...)dt + V_{wS}^q b dZ_t + V_{ww} \theta^{q\prime} \Gamma dZ_t \\ &= (...)dt + V_{wS}^q b dZ_t \\ &+ V_{ww}^q \left(\frac{(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1})}{-V_{ww}^q / V_w^q} - \frac{(\Gamma\Gamma')^{-1}\Gamma b' V_{Sw}^q}{V_{ww}^q} \right)' \Gamma dZ \end{aligned}$$

And therefore

$$\begin{aligned} \mathrm{cov}\left(\Gamma dZ, \frac{du'(c)}{u'(c)}\right) &= \left(\Gamma b' \frac{V_{wS}^q}{V_w^q}\right) dt - \Gamma\Gamma'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}) \\ &\quad - \Gamma\Gamma'(\Gamma\Gamma')^{-1}\Gamma b' \frac{V_{wS}^q}{V_w^q} dt \\ &= -(\mu - r\mathbf{1})dt, \end{aligned}$$

Consumption Capital Asset Pricing Model (CCAPM)

$$\mu - r\mathbf{1} = -\frac{1}{dt}\mathrm{cov}\left(\Gamma dZ, \frac{du'(c)}{u'(c)}\right)$$

and because $du'(c)/u'(c) = (...)dt + (u''(c)/u'(c))dc/c$, we have that

$$\mu - r\mathbf{1} = \frac{1}{-cu''(c)/u'(c)}\frac{1}{dt}\mathrm{cov}\left(\Gamma dZ, \frac{dc}{c}\right)$$

We derived this at the individual level, and there are different paths for extending it to the level of aggregate consumption. One is to restrict to the problem without state variables and assuming identical agents with log or power utility and different levels of wealth.

This theory is simple and appealing, but actually it doesn't work at all empirically. There are some desperate attempts to save it (e.g. long-term risk), but I don't recommend them.

Intertemporal Capital Asset Pricing Model (ICAPM)

The ICAPM says that in addition to the market, there could be pricing of changes in future investment opportunities. From the model with state variables, we can write the first-order condition for optimal portfolio choice as

$$\frac{1}{-V^q_{ww}/V^q_w}(\mu - r\mathbf{1}) = \Gamma\Gamma'\theta^q + \Gamma b'\frac{V^q_{Sw}}{V^q_{ww}}$$

Summing across agents q and dividing by Q, we have

$$\frac{\Sigma_q(1/(-V^q_{ww}/V^q_w))}{Q}(\mu - r\mathbf{1}) = \Gamma\Gamma'\frac{\Sigma_q\,\theta}{Q} + \Gamma b'\left(\frac{1}{Q}\sum_q\frac{V^q_{Sw}}{V^q_{ww}}\right)$$

or

$$\mu - r\mathbf{1} = \left((1/Q)\sum_q(1/(-V^q_{ww}/V^q_w))\right)^{-1}\left(\Gamma\Gamma'\theta^m + \Gamma b'\left(\frac{1}{Q}\sum_q\frac{V^q_{Sw}}{V^q_{ww}}\right)\right),$$

so that excess returns depend on covariance with the market (as in the CAPM) and covariances with the state variables.

Arbitrage Pricing Theory (APT)

The APT, motivated by absence of arbitrage, says that systemic risk that is common across assets can priced and idiosyncratic risk is not. This is the best model for thinking about the factor models (starting with Chen, Roll, and Ross (1986)[2] and Fama and French (1992)[3]). In the APT, asset returns are assumed to follow a K-factor model

$$\mu dt + (B|D)dZ_t,$$

where B is an $N \times K$ matrix of loadings on the common factors (the first K elements of dZ_t), and D is an $N \times N$ matrix giving the standard deviations of the error terms (the idiosyncratic noise). The APT says that the factor exposures are priced, but not the idiosyncratic noise:

$$\mu - r\mathbf{1} = B\gamma,$$

where γ $K \times 1$ is the vector of factor premia. The APT is the foundation of the popular factor investing models.

[2] Chen, Nai-Fu; Roll, Richard; Ross, Stephen A.; Journal of Business, July 1986, v. 59, iss. 3, pp. 383-403

[3] Fama, Eugene F.; French, Kenneth R.; Journal of Finance, June 1992, v. 47, iss. 2, pp. 427-65