Problem Set 5: Nontraded wealth
FIN 539 Mathematical Finance
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1. What is the difference between priced risk in the Intertemporal CAPM (ICAPM) and the Arbitrage Pricing Theory (APT)?

In the ICAPM, the priced factors are related to future investment opportunities, while in the APT, they can be any common factors across stocks.

2. Consider the problem of an agent who has labor income y_t that is not completely spanned by the single risky asset in the economy. We have the choice problem:

Given w_0 and y_0,
choose adapted portfolio θ_t, consumption c_t, and wealth w_t to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t}u(c_t)dt]$ (objective function)
subject to:
$(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_{1t}) - c_t dt + y_t dt)$ (budget constraint),
$(\forall t)(dy_t = ay_t dt + by_t dZ_t)$ (income dynamics),
and
$(\forall t)w_t \geq 0$ (no borrowing).

In this problem, Z_t is 2-dimensional and so is the constant vector b. The first element Z_{1t} of Z_t drives the stock price and can be hedged in the market, while the second element Z_{2t} does not affect stock prices and cannot be hedged in the market. Labor income is affected by both sources of risk. The portfolio θ_t, consumption c_t, wealth w_t, and income y_t are all 1-dimensional. The parameters ρ, r, μ, σ, and a are all constant real numbers.

A. Write down the martingale M_t for this problem.

$$M_t \equiv \int_{s=0}^{t} e^{-\rho s}u(c_s)ds + e^{-\rho t}V(w_t, y_t)$$

B. What does M_t represent given the optimal policies for portfolio, consumption, and wealth?

the conditional expectation at t of the objective if we always follow the optimal policies

What does M_t represent given a suboptimal policy?

the conditional expectation at t of the objective from following the suboptimal policy until t and the optimal policy from then on

For $t > s$, what is $E[M_s] - E[M_t]$?

the reduction of the objective function from following this policy instead of the optimal policy from time s to time t.

C. Derive the Bellman equation for this problem. The state variable vector is $X_t = (w_t, y_t)$, and we have that

$$dX_t = \left(\begin{array}{c} rw + \theta(\mu - r) - c + y \\ ay \end{array} \right) dt + \left(\begin{array}{cc} \theta\sigma & 0 \\ b_1 y & b_2 y \end{array} \right) dZ_t$$

and therefore

$$\begin{aligned}
\frac{\mathrm{E}[dM]}{e^{-\rho t}dt} &= u(c) - \rho V + (rw + \theta(\mu - r) - c + y)V_w + ayV_y \\
&\quad + \frac{1}{2}\mathrm{tr}\left(\left(\begin{array}{cc} \theta\sigma & 0 \\ b_1 y & b_2 y \end{array} \right) \left(\begin{array}{cc} \theta\sigma & b_1 y \\ 0 & b_2 y \end{array} \right) \left(\begin{array}{cc} V_{ww} & V_{wy} \\ V_{yw} & V_{yy} \end{array} \right) \right) \\
&= u(c) - \rho V + (rw + \theta(\mu - r) - c + y)V_w + ayV_y \\
&\quad + \frac{1}{2}\mathrm{tr}\left(\left(\begin{array}{cc} \theta^2\sigma^2 & \theta\sigma b_1 y \\ \theta\sigma b_1 y & (b_1^2 + b_2^2)y^2 \end{array} \right) \left(\begin{array}{cc} V_{ww} & V_{wy} \\ V_{yw} & V_{yy} \end{array} \right) \right) \\
&= u(c) - \rho V + (rw + \theta(\mu - r) - c + y)V_w + ayV_y \\
&\quad + \frac{1}{2}\mathrm{tr}\left(\left(\begin{array}{cc} \theta^2\sigma^2 V_{ww} + \theta\sigma b_1 y V_{yw} & \theta^2\sigma^2 V_{wy} \theta\sigma b_1 y V_{yy} \\ \theta\sigma b_1 y V_{ww} + (b_1^2 + b_2^2)y^2 V_{yw} & \theta\sigma b_1 y V_{wy} + (b_1^2 + b_2^2)y^2 V_{yy} \end{array} \right) \right) \\
&= u(c) - \rho V + (rw + \theta(\mu - r) - c + y)V_w + ayV_y \\
&\quad + \frac{1}{2}(\theta^2\sigma^2 V_{ww} + 2\theta\sigma b_1 y V_{yw} + (b_1^2 + b_2^2)y^2 V_{yy})
\end{aligned}$$

Therefore, the Bellman equation is

$$\begin{aligned}
0 \quad = \quad & \max_{c,\theta}(u(c) - \rho V + (rw + \theta(\mu - r) - c + y)V_w + ayV_y \\
& + \theta^2\sigma^2 V_{ww}/2 + \theta\sigma b_1 y V_{yw} + (b_1^2 + b_2^2)y^2 V_{yy}/2)
\end{aligned}$$

D. Solve for optimal c and θ in terms of derivatives of V.

The terms with c are the usual $u(c) - cV_w$, so we have the usual first-order ccondition $u'(c) = V_w$ and $c = I(V_w)$, where $I(\cdot)$ is the inverse function of $u'(c)$. The terms with θ are

$$\theta(\mu - r)V_w + \theta^2\sigma^2 V_{ww}/2 + \theta\sigma b_1 y V_{yw},$$

so the first-order condition is

$$0 = (\mu - r)V_w + \theta\sigma^2 V_{ww} + \sigma b_1 y V_{yw},$$

and the optimal portfolio is

$$\theta^* = \frac{\mu - r}{-\sigma^2 V_{ww}/V_w} - \frac{b_1 y V yw}{\sigma V_{ww}}$$