Problem Set 4: Multi-asset portfolio problem
FIN 539 Mathematical Finance
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1. **Homotheticity** Consider the log felicity (or utility) function $u(c) = \log(c)$. Then we will study variations of the following multi-asset optimization problem:

Given w_0,
choose adapted risky asset proportions θ_t, consumption c_t, and wealth w_t, to maximize $E[\int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt]$ (expected utility of lifetime consumption)
subject to:
$dw_t = rw_t dt + w_t \theta_t'((\mu - r\mathbf{1})dt + \Gamma dZ_t) - c_t dt$ (budget constraint)
$w_t \geq 0$ (no borrowing)

The choice variables are three processes: the vector of risky asset proportions $\theta_t \in \Re^N$, real-valued consumption c_t, and real-valued wealth w_t. The constant ρ is the pure rate of time discount, the constant r is the instantaneous riskfree rate of interest, $\mu \in \Re^N$ is the constant vector of mean risky asset returns, Γ is the constant $N \times k$ matrix of loadings of the returns on the different risks, and $\mathbf{1}$ is the N-vector of 1's. Assume the local covariance $\Gamma\Gamma'$ of returns is positive definite, and that there is at least one asset n with $\mu_n > r$.

A. Show that the form of the value function for this problem is $V(w) = v + \log(w)/\rho$ for some constant v.

B. Does the result in part A hold (perhaps for a different constant v) if we add the constraint

$$(\forall i, t)(0 \leq \theta_{it} \leq K_i)$$

where each $K_i > 0$ is a given constant? Explain why or why not. (If not, it suffices to show where the usual argument breaks down.)

C. Does the result in part A hold (perhaps for a different constant v) if we add the constraint

$$(\forall t)(0 \leq \theta_t \leq w_t K)$$

where $K > 0$ is a given constant. Explain why or why not. (If not, it suffices to show that the usual argument breaks down.)

2. **Bellman Equation** Consider the optimization problem in Problem 1, without either constraint described in Part B or Part C. (Note: this problem can be solved even if you did not solve Problem 1.)

A. Write down the process M_t for this problem.

B. What does M_t represent given the optimal policies for portfolio, consumption, and wealth? What does M_t represent given suboptimal policies? For $t < s$, what is $E[M_t] - E[M_s]$?

C. Derive the Bellman equation for this problem.

D. Solve for optimal c_t and θ_t in terms of derivatives of V.

E. From Problem 1, we can write the value function in the form $V(w) = v + \log(w)/\rho$. Using this formula, solve for the optimal c_t and θ_t in terms of w_t and the parameters.

F. Substitute the optimal portfolio and consumption policies into the Bellman equation, and solve the optimized Bellman equation for v.