Problem Set 3: Homotheticity and Bellman Equation
FIN 539 Mathematical Finance
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1. **Homotheticity** Consider the HARA (Hyperbolic Absolute Risk Aversion) felicity (or utility) function $u(c) = (c - \underline{c})^{1-R}/(1-R)$, where $\underline{c}$ is the subsistence consumption (the minimal consumption needed to survive) and $R > 0$, $R \neq 1$, is the relative risk aversion for the increase of consumption above the subsistence level. Then we will study the following optimization problem:

Given w_0,
choose portfolio θ_t, consumption c_t, and wealth w_t to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt]$ (expected utility of lifetime consumption)
subject to:
$dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t) - c_t dt$ (budget constraint)
$(\exists K \in \Re)(\forall t) w_t \geq -K$ (limited borrowing)

Prove that the value function for this problem is of the form $V(w) \equiv (w - \underline{c}/r)^{1-R} v$ for some constant v. Assume that $w_0 > \underline{c}/r$ so there is enough wealth to pay for subsistence consumption with something left over. (Hint: note that wealth of $\underline{c}/r$ can be thought of as being committed to funding the subsistence consumption $\underline{c}$ so "free" or "discretionary" wealth at t is $w_t - \underline{c}$. Define new choice variables: portfolio choice $\hat{\theta} \equiv \theta_t/(w_0 - \underline{c}/r)$ normalized by initial free wealth, consumption $\hat{c}_t \equiv (c_t - \underline{c})/(w_0 - \underline{c}/r)$ in excess of the subsistence consumption, normalized by initial free wealth, and free wealth $\hat{w}_t \equiv (w_t - \underline{c}/r)/(w_0 - \underline{c}/r)$ normalized by initial free wealth.)

Define $\hat{\theta} \equiv \theta_t/(w_0 - \underline{c}/r)$, $\hat{c}_t \equiv (c_t - \underline{c})/(w_0 - \underline{c}/r)$, and $\hat{w}_t \equiv (w_t - \underline{c}/r)/(w_0 - \underline{c}/r)$, and let $\hat{K} \equiv (K + \underline{c}/r)/(w_0 - \underline{c}/r)$. Then we can rewrite the choice problem as:

Given w_0,
choose $\hat{\theta}_t$, $\hat{c}_t$, and $\hat{w}_t$ to
maximize $(w_0 - \underline{c}/r)^{1-R} E[\int_{t=0}^{\infty} e^{-\rho t} \frac{\hat{c}_t{}^{1-R}}{1-R} dt]$
subject to:
$d\hat{w}_t = r\hat{w}_t dt + \hat{\theta}_t((\mu - r)dt + \sigma dZ_t) - \hat{c}_t dt$
$(\exists \hat{K} \in \Re)(\forall t) \hat{w}_t \geq -\hat{K}$.

Note that w_0 does not appear except in the positive factor $(w_0 - \underline{c}/r)^{1-R}$ multiplying the objective function. Therefore, this factor does not affect the optimal choice but only multiplies the value by this factor. Therefore, the value of the problem is $(w_0 - \underline{c}/r)^{1-R}v$, where v is the optimized value of $E[\int_{t=0}^{\infty} e^{-\rho t} \frac{\hat{c}_t{}^{1-R}}{1-R} dt]$, which is the value of the problem when $(w_0 - \underline{c}/r)^{1-R} = 1$.

2. **Bellman Equation** Consider the optimization problem in Problem 1. (Note: this problem can be solved even if you did not solve Problem 1.)

A. Write down the martingale M_t for this problem.

$M_t \equiv \int_{s=0}^{t} e^{-\rho s} \frac{(c_s - \underline{c})^{1-R}}{1-R} ds + e^{-\rho t} V(w_t)$

B. What does M_t represent given the optimal policies for portfolio, consumption, and wealth?

the conditional expectation at t of the integral defining the objective function if we always follow the optimal policies

What does M_t represent given suboptimal policies?

the conditional expectation at t of the integral defining the objective function if we follow the suboptimal policies until t and the optimal policies from then on

For $t > s$, what is $E[M_s] - E[M_t]$?

the loss, in units of change of the objective function, from following this policy from time s to time t, compared with following the optimal policy. The loss is 0 for the optimal policy.

C. Derive the Bellman equation for this problem.

$$
\begin{aligned}
dM_t &= e^{-\rho t} \frac{(c_t - \underline{c})^{1-R}}{1-R} dt - \rho e^{-\rho t} V(w_t) dt + e^{-\rho t} V'(w_t) dw_t \\
&\quad + \frac{1}{2} e^{-\rho t} V''(w_t) (dw_t)^2
\end{aligned}
$$

$$= \quad e^{-\rho t}\left(\frac{(c_t-\underline{c})^{1-R}}{1-R} - \rho V(w_t) + (rw_t + \theta_t(\mu - r) - c_t)V'(w_t) \right.$$
$$\left. + \frac{1}{2}V''(w_t)\theta_t^2\sigma^2\right) dt + V'(w_t)\theta_t\sigma dZ_t$$

Now the drift $E[dM]/dt$ is zero for the optimal strategy and zero or negative for all strategies. Therefore, the maximum drift over all choices of c_t and θ_t is equal to zero, which is the Bellman equation.

$$\max_{c,\theta}\left(\frac{(c-\underline{c})^{1-R}}{1-R} - \rho V + (rw + \theta(\mu - r) - c)V' + \frac{\theta^2\sigma^2}{2}V''\right) = 0$$

D. Solve for optimal c_t and θ_t in terms of derivatives of V, and substitute the optimized values into the Bellman equation.

FOC for c:

$$(c-\underline{c})^{-R} - V' = 0$$

so

$$c^* = \underline{c} + (V')^{-1/R}$$

FOC for θ:

$$(\mu - r)V' + \theta\sigma^2 V'' = 0$$

so

$$\theta^* = -\frac{\mu - r}{\sigma^2}\frac{V'}{V''}$$

Optimized Bellman equation:

$$\frac{R(V')^{1-1/R}}{1-R} - \rho V + r(w - \underline{c}/r)V' - \frac{(\mu - r)^2}{2\sigma^2}\frac{(V')^2}{V''} = 0$$

E. From Problem 1, we can write the value function in the form $V(w) = \frac{k}{1-R}(w - \underline{c}/r)^{1-R}$, where $k = (1-R)v$. Using this formula, solve for the optimal c_t and θ_t in terms of w_t and the parameters.

$$V(w) = \frac{k}{1-R}(w - \underline{c}/r)^{1-R}$$

$$V'(w) = k(w - \underline{c}/r)^{-R}$$

$$V''(w) = -Rk(w - \underline{c}/r)^{-R-1}$$

$$c^* = \underline{c} + (V')^{-1/R} = \underline{c} + (w - \underline{c}/r)k^{-1/R}$$

$$\begin{aligned}
\theta^* &= -\frac{\mu - r}{\sigma^2}\frac{V'}{V''} = -\frac{\mu - r}{\sigma^2}\frac{(w - \underline{c}/r)^{-R}k}{-R(w - \underline{c}/r)^{-R-1}k} \\
&= \frac{\mu - r}{\sigma^2 R}(w - \underline{c}/r)
\end{aligned}$$

F. Solve the Bellman equation for k.

$$\begin{aligned}
0 &= \frac{R(V')^{1-1/R}}{1-R} - \rho V + r(w - \underline{c}/r)V' - \frac{(\mu - r)^2}{2\sigma^2}\frac{(V')^2}{V''} \\
&= \frac{(w - \underline{c}/r)^{1-R}k}{1-R}\left(Rk^{-1/R} - \rho + (1-R)r + \frac{(\mu - r)^2}{2\sigma^2}\frac{1-R}{R}\right)
\end{aligned}$$

$$k = \left(\frac{\rho}{R} + \left(1 - \frac{1}{R}\right)r + \left(1 - \frac{1}{R}\right)\frac{(\mu - r)^2}{2\sigma^2 R}\right)^{-R}$$