Problem Set 1: Kuhn-Tucker conditions
FIN 539 Mathematical Finance
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1. A complete-markets portfolio choice problem.

Given initial wealth w_0,
choose state-contingent consumptions c_1, ... c_Ω, to
maximize $\mathrm{E}[u(c_\omega)]$ (objective function)
st $\mathrm{E}[\xi_\omega c_\omega] = w_0$ (budget constraint)
$(\forall\omega) c_\omega \geq \bar{c}$ (consumption floor).

In this problem, $\mathrm{E}[u(c)]$ is the von Neumann-Morgenstern utility function with $u'(c) > 0$ and $u''(c) < 0$, $\omega = 1, ..., \Omega$ are the states of nature, ξ is the vector of state-price densities (or stochastic discount factors), and $\bar{c}$ is an exogenously-imposed floor on consumption.

A. What are the Kuhn-Tucker (KT) conditions for the problem?

B. Show that the solution to the Kuhn-Tucker conditions is given by

$$c_\omega = \max(I(\lambda\xi_\omega), \bar{c}),$$

where $I(z)$ is the inverse function of the marginal utility $u'(c)$ and λ is the Lagrangian multiplier on the budget constraint.

C. Suppose that $u(c) = \sqrt{k_1^2 + 4k_2 c} + k_1 \log(\sqrt{k_1^2 + 4k_2 c} - k_1)$; this is a special case of GOBI preferences.[1] Then show that

$$I(z) = \frac{k_1}{z} + \frac{k_2}{z^2}$$

D. Given the choice of the utility in part C, write down the form of consumption.

[1] Dybvig, Philip H., and Fang Liu, 2018, On Investor Preferences and Mutual Fund Separation, Journal of Economic Theory 174, 224–260.

E. Write down the equation that should be solved for the Lagrange multiplier.