Mathematical Finance Mini Exam Answers, Spring A 2022
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I. Short answer (30 points).

A. In words, what does it mean when an agent has constant relative risk aversion?

At all initial wealth levels, the agent has the same preferences over gambles proportional to initial wealth.

B. What is the difference between priced risk in the Capital Asset Pricing Model (CAPM) and the Intertermporal Capital Asset Pricing Model (ICAPM)?

In the CAPM, only market risk is priced, while in the ICAPM state variables predicting future investment opportunities can also be priced.

C. Why is it a problem if a client gives you a covariance matrix of returns to use that has mostly positive eigenvalues but a few negative ones? How can you fix it?

If there is any negative eigenvalue, the matrix cannot be a covariance matrix because some portfolio will have a negative variance. This makes it impossible to simulate returns, and tends to give extreme results or no solution in a portfolio problem. The covariance matrix can be repaired by replacing the negative eigenvalues by small positive numbers.

II. Bellman Equation (40 points) Consider a continuous-time portfolio choice problem with power utility $u(w_T) = w_T^{1-R}/(1-R)$ for consumption at the terminal horizon $T > 0$, where $R > 0$, $R \neq 1$. There is a constant riskfree rate $r > 0$ and a single risky asset with expected return $\mu > r$ per unit time and local variance σ^2 per unit time. Then the choice problem is

Given w at time 0,
choose adapted θ_t and w_t to
maximize $\mathrm{E}[e^{-\rho T}\frac{w_T^{1-R}}{1-R}]$
s.t. $w_0 = w$

$(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t))$
$(\forall t)(w_t \geq 0)$

A. For $t < T$, what is the process M_t for this problem?

$M_t = e^{-\rho t} V(w_t, t)$

B. What does M_t represent given the optimal policies for the portfolio and wealth? What does M_t represent given an arbitrary policy? For $s < t$, what is $\mathrm{E}[M_s] - \mathrm{E}[M_t]$?

For the optimal policy, M_t is the conditional expectation at t of the realied objective function when following the optimal policy. For an arbitrary policy, M_t is the conditional expectation at t of the realied objective function when following the arbitrary policy until t and the optimal policy from then on. $\mathrm{E}[M_s] - \mathrm{E}[M_t]$ is the cost in terms of the objective function of any mistakes made between time s and time t.

C. Derive the Bellman equation for this problem.

For the optimal strategy, M_t is a martingale and therefore $E[dM_t] = 0$. For an arbitrary strategy M_t is a supermartingale and therefore $E[dM_t] \leq 0$. These imply that $\max_\theta E[dM_t] = 0$. Now we can compute $E[dM_t]$ using Itô's lemma:

$$\frac{E[dM_t]}{e^{-\rho t} dt} = -\rho V + V_t + (rw + \theta(\mu - r))V_w + \frac{1}{2}\theta^2\sigma^2 V_{ww}$$

so the Bellman equation is

$$\max_\theta(-\rho V + (rw + \theta(\mu - r))V_w + \frac{1}{2}\theta^2\sigma^2 V_{ww}) = 0$$

D. Solve for optimal θ in terms of derivatives of V.

$$(\mu - r)V_w + \sigma^2\theta V_{ww} = 0$$
$$\theta = \frac{\mu - r}{\sigma^2}\frac{1}{-V_{ww}/V_w}$$

E. A scaling (homotheticity) argument can prove that the value function of this problem has the form $V(w_t, t) = v(t)w_t^{1-R}/(1-R)$. (Don't prove this!) Given this information, derive the optimal θ as a function of w.

$$V(w,t) = v(t)\frac{w^{1-R}}{1-R}$$
$$V_w(w,t) = v(t)w^{-R}$$
$$V_{ww}(w,t) = -Rv(t)w^{-R-1}$$
$$-V_{ww}/V_w = R/w$$
$$\theta = \frac{\mu - r}{\sigma^2}\frac{1}{-V_{ww}/V_w} = \frac{\mu - r}{\sigma^2 R}w$$

III. One-shot approach (30 points) Consider a continuous time portfolio choice problem with log utility $\log(w_T)$ for consumption at the terminal horizon $T > 0$. There is a constant riskfree rate $r > 0$ and a single risky asset with expected return $\mu > r$ per unit time and local variance σ^2 per unit time. Then the choice problem is

Given w at time 0,
choose w_T to
maximize $E[u(w_T)]$
s.t. $E[\xi_T w_T] = w_0$

where $\xi_t = \exp((-r - \kappa^2/2)t - \kappa Z_t)$, as derived in class, where $\kappa \equiv (\mu - r)/\sigma$ and the standard Wiener process Z_t is the uncertainty driving the stock price.

A. What is the first-order condition for the optimum? Write w_T as a function of ξ_T and the Lagrangian multiplier (λ).

$$u'(w_T) = \lambda\xi_T$$
$$u'(w) = 1/w \Rightarrow w_T = \frac{1}{\lambda\xi_T}$$

B. Solve for λ and write w_T as a function of w_0 and ξ_T.

$$w_0 = E[\xi_T w_T] = E[\xi_T \frac{1}{\lambda \xi_T}] = \frac{1}{\lambda}$$

$$\lambda = 1/w_0$$

$$w_T = \frac{w_0}{\xi_T}$$

C. Write w_t in terms of w_T ξ_t, and ξ_T.

$$w_t = E_t[\frac{\xi_T}{\xi_t} w_T]$$

D. Compute w_t as a function of w_0 and ξ_t.

$$w_t = E_t[\frac{\xi_T}{\xi_t} \frac{w_0}{\xi_T}] = E_t[\frac{w_0}{\xi_t}] = \frac{w_0}{\xi_t}$$

IV. Challenger (10 bonus points) [This is hard: don't work on this problem until you have completed and checked everything else.] Consider our standard infinite-horizon problem with fixed coefficients, a single risky asset, and constant relative risk aversion R. Fix R, $0 < R < 1$. Then the problem is

Given w_0 at time 0,
choose adapted θ_t, c_t, and w_t to
maximize $\mathrm{E}\left[\int_{t=0}^{\infty} e^{-\rho t} \frac{c_t^{1-R}}{1-R}\right]$
s.t. $(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t)) - c_t dt$
$(\forall t)(w_t \geq 0)$

For what values of the parameters μ, σ, r, R, and ρ does the problem have a solution? Explain the economics of your result; prove your claim for full credit.

Some formulas that might be useful

univariate Itô's lemma:

Let $dX_t = a_t dt + \sigma_t dZ_t$ where Z is a standard Wiener process, and let $f(X,t)$ have continuous partial derivatives f_X, f_{XX}, and f_t. Then

$$df(X_t,t) = f_X(X_t,t)(a_t dt + \sigma_t dZ_t) + f_t(X_t,t)dt + \frac{\sigma_t^2}{2} f_{XX}(X_t,t)dt.$$

multivariate Itô's lemma:

Let $H : \Re^d \times [0,T] \to \Re$ with continuous partial derivatives $H_x(x,t)$, $H_{xx}(x,t)$, and $H_t(x,t)$. Let $dX_t = g(t)dt + G(t)dZ_t$, where Z_t is an m-dimentional standard Wiener process. Then $Y_t \equiv H(X_t,t)$ is an Itô process with stochastic differential

$$dY = H_t dt + H_x dX + \frac{1}{2}\mathrm{tr}(GG' H_{xx})dt$$

Note: if H takes values in $\Re^K$, we can apply the result elementwise.

Black-Scholes differential equation:

$$0 = -r\mathcal{O} + \mathcal{O}_t + rS\mathcal{O}_S + \frac{\sigma^2}{2} S^2 \mathcal{O}_{SS},$$

State-price density (stochastic discount factor) if markets are complete:

Let security 0 have a riskless mean return r and any other asset $n = 1, ..., N$ has re-invested risky return $dS_{nt}/S_{nt} = \mu_{nt} t dt + \gamma_{nt} dZ_t$.

$$d\xi = -rdt - (\mu - r\mathbf{1})'(\Gamma')^{-1} dZ_t$$

where

$$\Gamma = (\gamma_1 | \gamma_2 | \dots | \gamma_N)'.$$

Univariate state-price density:

$$d\xi_t/\xi_t = -rdt - \kappa dZ_t,$$

where $\kappa \equiv (\mu - r)/\sigma$, and with constant coefficients and taking $\xi_0 = 1$ wlog, we have

$$\xi_t = \xi_0 \exp((-r - \kappa^2/2)t - \kappa Z_t),$$

Normal moment generating function:

If $x \sim N(m, s)$, $\mathrm{E}[e^x] = e^{m+s^2/2}$

Arrow-Pratt coefficient of absolute risk aversion:

$\frac{-u''(c)}{u'(c)}$

Arrow-Pratt coefficient of relative risk aversion:

$\frac{-cu''(c)}{u'(c)}$

Constant Absolute Risk Aversion (CARA) utility with risk aversion $A > 0$:

$$u(c) = -\frac{\exp(-Ac)}{A}$$

Constant Relative Risk Aversion (CRRA) utility with risk aversion $R > 0$:

$$u(c) = \begin{cases} \frac{c^{1-R}}{1-R} & \text{for } R \neq 1 \\ \log(c) & \text{for } R = 1 \end{cases}$$

Kuhn-Tucker conditions:

For the optimization model

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$,

the Kuhn-Tucker conditions are

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$