Mathematical Finance Mini Exam, Spring A 2021
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This is a closed-book exam: you may not use any books, notes, or electronic devices (calculators, headphones, laptops, etc.), except for the Zoom session for proctoring (camera must be on), reading the exam, asking me questions, and for submitting your exam. Mark your answers on paper and submit pictures of your answer sheets in Canvas.

There are no trick questions on the exam, but you should read the questions carefully.

PLEDGE (required)

The work on this exam will be mine alone, and I will conform with the rules of the exam and the code of conduct of the Olin Business School.

Signed name ________________________________

Good luck!

I. Short answer (30 points).

A. State in words the Fundamental Theorem of Asset Pricing (FTAP).

B. What is the difference between priced risk in the Capital Asset Pricing Model (CAPM) and the APT (Arbitrage Pricing Theory)?

C. If a client gives you a covariance matrix of returns to use that has mostly positive eigenvalues but a few negative ones, is that a problem? Why?

II. Homotheticity (30 points) Consider a continuous-time portfolio choice problem with log utility $u(W_T) = e^{-\rho T}\log(W_T)$ for consumption at the terminal horizon $T > 0$. There is a constant riskfree rate $r > 0$ and a single risky asset with expected return $\mu > r$ per unit time and local variance σ^2 per unit time. Then the choice problem is

Given w,
choose adapted θ_t and w_t to
maximize $\mathrm{E}[e^{-\rho T}\log(w_T)]$
s.t. $w_0 = w$
$(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t))$
$(\forall t)(w_t \geq 0)$

A. Write down the choice problem whose optimized value is the objective function $V(w,t)$.

B. Prove that there exists some function $v(\cdot)$ such that $V(w,t) = v(t) + e^{-\rho(T-t)}\log(w)$.

C. Would the result in part B still be true (perhaps for a different function $v(t)$) if we added the constraint $(\forall s)\theta_s \leq \bar{\theta} w_s$ for some positive constant $\bar{\theta}$? Explain why it is still true or why the proof fails.

III. Bellman equation (40 points) Consider the portfolio choice problem from question II. (Note: you do not have to solve Problem II first to solve this problem.)

Given w_0 at time 0,
choose adapted θ_t and w_t to
maximize $\mathrm{E}[e^{-\rho T}\log(W_T)]$

s.t. $(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t))$
$(\forall t)(w_t \geq 0)$

A. Write down the process M_t for this problem.

B. What does M_t represent given the optimal policies for the portfolio and wealth? What does M_t represent given an arbitrary policy? For $t > s$, what is $\mathrm{E}[M_s] - \mathrm{E}[M_t]$?

C. Derive the Bellman equation for this problem.

D. Solve for optimal θ_t in terms of derivatives of V.

E. In question II, it is proven that the value function of this problem has the form $V(w_t, t) = v(t) + e^{-\rho(T-t)} \log(w_t)$. Given this information, derive the optimal θ as a function of w.

IV. Challenger (10 bonus points) Consider our standard infinite-horizon problem with fixed coefficients and constant relative risk aversion R. Fix R, $0 < R < 1$. Then the problem is

Given w_0 at time 0,
choose adapted θ_t, c_t, and w_t to
maximize $\mathrm{E}\left[\int_{t=0}^{\infty} e^{-\rho t} \frac{c_t^{1-R}}{1-R}\right]$
s.t. $(\forall t)(dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t)) - c_t dt$
$(\forall t)(w_t \geq 0)$

For what values of the parameters μ, σ, r, R, and ρ does the problem have a solution? Explain the economics of your result; prove your claim for full credit. (This is hard: don't work on this problem until you have completed and checked everything else.)

Some formulas that might be useful

univariate Itô's lemma:

Let $dX_t = a_t dt + \sigma_t dZ_t$ where Z is a standard Wiener process, and let $f(X,t)$ have continuous partial derivatives f_X, f_{XX}, and f_t. Then

$$df(X_t,t) = f_X(X_t,t)(a_t dt + \sigma_t dZ_t) + f_t(X_t,t)dt + \frac{\sigma_t^2}{2} f_{XX}(X_t,t)dt.$$

multivariate Itô's lemma:

Let $H : \Re^d \times [0,T] \to \Re$ with continuous partial derivatives $H_x(x,t)$, $H_{xx}(x,t)$, and $H_t(x,t)$. Let $dX_t = g(t)dt + G(t)dZ_t$, where Z_t is an m-dimentional standard Wiener process. Then $Y_t \equiv H(X_t,t)$ is an Itô process with stochastic differential

$$dY = H_t dt + H_x dX + \frac{1}{2}\mathrm{tr}(GG' H_{xx})dt$$

Note: if H takes values in $\Re^K$, we can apply the result elementwise.

Black-Scholes differential equation:

$$0 = -r\mathcal{O} + \mathcal{O}_t + rS\mathcal{O}_S + \frac{\sigma^2}{2} S^2 \mathcal{O}_{SS},$$

State-price density (stochastic discount factor) if markets are complete:

Let security 0 have a riskless mean return r and any other asset $n = 1, ..., N$ has re-invested risky return $dS_{nt}/S_{nt} = \mu_{nt} t dt + \gamma_{nt} dZ_t$.

$$d\xi = -r dt - (\mu - r\mathbf{1})'(\Gamma')^{-1} dZ_t$$

where

$$\Gamma = (\gamma_1|\gamma_2|\ldots|\gamma_N)'.$$

Univariate state-price density:

$$d\xi_t/\xi_t = -rdt - \kappa dZ_t,$$

where $\kappa \equiv (\mu - r)/\sigma$, and with constant coefficients and taking $\xi_0 = 1$ wlog, we have

$$\xi_t = \xi_0 \exp((-r - \kappa^2/2)t - \kappa Z_t),$$

Normal moment generating function:

If $x \sim N(m, s)$, $\mathrm{E}[e^x] = e^{m+s^2/2}$

Arrow-Pratt coefficient of absolute risk aversion:

$\frac{-u''(c)}{u'(c)}$

Arrow-Pratt coefficient of relative risk aversion:

$\frac{-cu''(c)}{u'(c)}$

Constant Absolute Risk Aversion (CARA) utility with risk aversion $A > 0$:

$$u(c) = -\frac{\exp(-Ac)}{A}$$

Constant Relative Risk Aversion (CRRA) utility with risk aversion $R > 0$:

$$u(c) = \begin{cases} \frac{c^{1-R}}{1-R} & \text{for } R \neq 1 \\ \log(c) & \text{for } R = 1 \end{cases}$$

Kuhn-Tucker conditions:

For the optimization model

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$,

the Kuhn-Tucker conditions are

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$