Problem Set 2: Bellman Preliminaries and Covariance Matrices
FIN 539 Mathematical Finance
P. Dybvig

Avantika and Jiaen will discuss this homework at the TA session, time TBA.

1. **Bellman Equation: preliminaries** This problem does some preliminary calculations for a problem we will solve in next week's homework.

Consider the HARA (Hyperbolic Absolute Risk Aversion) felicity (or utility) function $u(c) = (c - \underline{c})^{1-R}/(1 - R)$, where $\underline{c}$ is the subsistence consumption (the minimal consumption needed to survive) and $R > 0$, $R \neq 1$, is the relative risk aversion for the increase of consumption above the subsistence level. Then we will study the following optimization problem:

Given w_0,
choose portfolio θ_t, consumption c_t, and wealth w_t to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t} u(c_t)dt]$ (expected utility of lifetime consumption)
subject to:
$dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t) - c_t dt$ (budget constraint)
$(\exists K \in \Re)(\forall t) w_t \geq -K$ (limited borrowing)

A. The Bellman equation is derived from dM_t for a process M_t defined in class which gives the realized value of the objective at time t given we are following a possibly suboptimal strategy until time t and then switching to the optimal strategy from then on. One thing we will have to compute in deriving dM_t is $d(e^{-\rho t}V(w_t))$, where $V(w_t)$ is the value function (as yet unknown, but assumed to be twice continuously differentiable) and dw_t is given by the budget constraint in the problem above. Use Itô's lemma to derive $d((e^{-\rho t}V(w_t))$.

B. Another term in deriving dM_t comes from taking a derivative of an integral with respect to parameters. This is ordinary calculus (Leibniz' rule), and the integral is done statewise. Compute $d(\int_{s=0}^{t} e^{-\rho s}u(c_s)ds)/dt$.

C. Optimization of c at a point of time maximizes an objective function that equals $u(c) - V_w c$ (where $u(c) = (c - \underline{c})^{1-R}/(1 - R)$) plus other terms that do not depend on c. Solve for the optimal c, and the maximized value of

$u(c) - V_w c$. Note: V_w does not depend on c.

D. Optimization of θ at a point in time maximizes an objective function that equals $\theta(\mu - r)V_w + \theta^2\sigma^2 V_{ww}/2$. Solve for the optimal θ and the maximized value of $\theta(\mu - r)V_w + \theta^2\sigma^2 V_{ww}/2$. Note: V_w and V_{ww} do not depend on θ.

2. **Positive definite covariance matrix** Suppose your client gives you the following 2×2 covariance matrix:

$$V = \begin{vmatrix} 0.0495 & 0.0505 \\ 0.0505 & 0.0495 \end{vmatrix}$$

(Okay, your client is more likely to give you a defective 10×10 covariance matrix, but I want this to be easy enough to solve by hand.)

A. Compute the eigenvalues and corresponding eigenvectors. (Hint: to solve for the eigenvalues, use the equation $\det(A - \lambda I) = 0$. Then use the eigenvalue equation $Ax_i = \lambda_i x_i$ to solve for the eigenvector.)

B. Show that V is not positive semi-definite.

C. Change any negative eigenvectors to 0.0001 and compute the new covariance matrix. (Hint: used the normalized eigenvectors and the formula $V = X'\Lambda X$.)