Problem Set 4: Multi-asset portfolio problem
FIN 539 Mathematical Finance
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Avantika and Jiaen will discuss this homework at the TA session, time TBA.

1. **Homotheticity** Consider the log felicity (or utility) function $u(c) = \log(c)$. Then we will study variations of the following multi-asset optimization problem:

Given w_0,
choose adapted risky asset proportions θ_t, consumption c_t, and wealth w_t, to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt]$ (expected utility of lifetime consumption)
subject to:
$dw_t = rw_t dt + w_t \theta_t'((\mu - r\mathbf{1})dt + \Gamma dZ_t) - c_t dt$ (budget constraint)
$w_t \geq 0$ (no borrowing)

The choice variables are three processes: the vector of risky asset proportions $\theta_t \in \Re^N$, real-valued consumption c_t, and real-valued wealth w_t. The constant ρ is the pure rate of time discount, the constant r is the instantaneous riskfree rate of interest, $\mu \in \Re^N$ is the constant vector of mean risky asset returns, Γ is the constant $N \times k$ matrix of loadings of the returns on the different risks, and $\mathbf{1}$ is the N-vector of 1's. Assume the local covariance $\Gamma\Gamma'$ of returns is positive definite, and that there is at least one asset n with $\mu_n > r$.

A. Show that the form of the value function for this problem is $V(w) = v + \log(w)/\rho$ for some constant v.

Let $\hat{\theta}_t \equiv \theta_t$ (no need to change since it is already normalized by wealth), $\hat{c}_t \equiv c_t/w_0$, and $\hat{w}_t \equiv w_t/w_0$. Then we can write the problem as

Given w_0,
choose adapted $\hat{\theta}_t$, $\hat{c}_t$, and $\hat{w}_t$ to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t} \log(w_0 \hat{c}_t) dt]$
subject to:
$d\hat{w}_t = r\hat{w}_t dt + \hat{w}_t \hat{\theta}_t'((\mu - r\mathbf{1})dt + \Gamma dZ_t) - \hat{c}_t dt$
$\hat{w}_t \geq 0$ (no borrowing)

The objective function can be written as

$$\begin{aligned} E[\int_{t=0}^{\infty} e^{-\rho t} \log(w_0 \hat{c}_t) dt] &= \int_{t=0}^{\infty} e^{-\rho t} \log(w_0) dt + E[\int_{t=0}^{\infty} e^{-\rho t} \log(\hat{c}_t) dt] \\ &= \log(w_0)/\rho + E[\int_{t=0}^{\infty} e^{-\rho t} \log(\hat{c}_t) dt]. \end{aligned}$$

Writing the problem this way, the constraints do not depend on w_0 and the objective function depends on w_0 only through the additive constant $\log(w_0)/\rho$. Therefore, the optimal choices of $\hat{\theta}_t$, $\hat{c}_t$, and $\hat{w}_t$ do not depend on w and we have that $V(w) = v + \log(w)/\rho$, where v is the optimized value of the second term, which is $V(1)$.

B. Does the result in part A hold (perhaps for a different constant v) if we add the constraint

$$(\forall i, t)(0 \leq \theta_{it} \leq K_i)$$

where each $K_i > 0$ is a given constant? Explain why or why not. (If not, it suffices to show where the usual argument breaks down.)

Yes, it does, since $\hat{\theta}_t = \theta_t$ and the new constraints in terms of the transformed variables do not depend on w.

C. Does the result in part A hold (perhaps for a different constant v) if we add the constraint

$$(\forall t)(0 \leq \theta_t \leq w_t K)$$

where $K > 0$ is a given constant. Explain why or why not. (If not, it suffices to show that the usual argument breaks down.)

No, because the constraint in the transformed variables becomes

$$(\forall t)(0 \leq \hat{\theta}_t \leq w \hat{w}_t K),$$

which does depend on w. In particular, this is very binding when w is very small, but not very binding when w is large. Since there is some asset with $\mu_i > r$, holding a small positive amount of that asset would dominate just holding the riskless asset, so the constraint is strictly binding for w sufficiently small.

2. **Bellman Equation** Consider the optimization problem in Problem 1, without either constraint described in Part B or Part C. (Note: this problem can be solved even if you did not solve Problem 1.)

A. Write down the process M_t for this problem.

$$M_t \equiv \int_{s=0}^{t} e^{-\rho s} \log(c_s) ds + e^{-\rho t} V(w_t)$$

B. What does M_t represent given the optimal policies for portfolio, consumption, and wealth? What does M_t represent given suboptimal policies? For $t < s$, what is $E[M_t] - E[M_s]$?

Given the optimal policy, M_t is the conditional expectation at t of the "realized utility" (the quantity in the expectation of the objective function) for the optimal strategy. Given a suboptimal strategy, it is the conditional expectation at t of following the suboptimal strategy until t and then the optimal strategy from then on. $E[M_t] - E[M_s]$ is the decrease in the objective function due to mistakes made between time t and time s.

C. Derive the Bellman equation for this problem.

Use Itô's lemma and the budget constraint to compute dM_t:

$$M_t = \int_{s=0}^{t} e^{-\rho s} \log(c_s) ds + e^{-\rho t} V(w_t),$$

we can compute

$$E\left[\frac{dM}{e^{-\rho t} dt}\right] = \log(c) - \rho V + (rw + w\theta'(\mu - r\mathbf{1}) - c)V_w$$

$$+\frac{1}{2}\mathrm{tr}(w\theta'\Gamma\Gamma'\theta w V_{ww}),$$

where $\theta'\Gamma\Gamma'\theta$, w, and V_{ww} are scalars, so the Bellman equation is

$$\max_{c,\theta}\left(\log(c) - \rho V + ((r + \theta'(\mu - r\mathbf{1}))w - c)V_w + \frac{w^2 V_{ww}}{2}\theta'\Gamma\Gamma'\theta\right) = 0$$

D. Solve for optimal c_t and θ_t in terms of derivatives of V.

The terms involving θ are

$$\theta'(\mu - r\mathbf{1})wV_w + \frac{w^2 V_{ww}}{2}\theta'\Gamma\Gamma'\theta,$$

where $\Gamma\Gamma'$ is the local covariance matrix of security returns. The first-order condition for optimal θ is

$$(\mu - r\mathbf{1})wV_w + w^2\Gamma\Gamma'\theta V_{ww} = 0$$

As before, $u'(c) = V_w$ so $c = I(V_w)$, but now the optimal portfolio is

$$\theta^* = \frac{1}{-wV_{ww}/V_w}(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1})$$

The optimization is locally a mean-variance problem. Note the coefficient of relative risk aversion of the value function in the denominator.

E. From Problem 1, we can write the value function in the form $V(w) = v + \log(w)/\rho$. Using this formula, solve for the optimal c_t and θ_t in terms of w_t and the parameters.

For some v,

$$V(w) = v + \log(w)/\rho$$

$$V_w(w) = \frac{1}{\rho w}$$

$$V_{ww}(w) = -\frac{1}{\rho w^2}$$

Now, $u(c) = \log(c)$, $u'(c) = 1/c$, and $I(z) = 1/z$. Therefore,

$$c_t^{\ *} = \frac{1}{V_w(w_t)} = \rho w$$

$$\begin{aligned}\theta_t^{\ *} &= \frac{1}{-wV_{ww}/V_w}(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}) \\ &= (\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}).\end{aligned}$$

F. Substitute the optimal portfolio and consumption policies into the Bellman equation, and solve the optimized Bellman equation for v.

$$\begin{aligned}0 &= \log(\rho w) - \rho(v + \log(w)/\rho) + ((r + (\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}))w \\ &\quad -\rho w)\frac{1}{\rho w} - \frac{w^2}{2\rho w^2}(\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}\Gamma\Gamma'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}) \\ &= \log(\rho) + \log(w) - \rho v - \log(w) + \frac{r - \rho}{\rho} \\ &\quad + \frac{1}{2\rho}(\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1}).\end{aligned}$$

Therefore,

$$v = \frac{1}{\rho}\left(\log(\rho) + \frac{r - \rho}{\rho} + \frac{1}{2\rho}(\mu - r\mathbf{1})'(\Gamma\Gamma')^{-1}(\mu - r\mathbf{1})\right)$$

Recall that this is the value function $V(1)$ when $w = 1$. The first term in parentheses is what we get if wealth never changes and we consume ρ forever. The second term is the value of consumption changing over time because the interest rw funds more or less than the consumption ρw. The third term in parentheses is the value of investing in the market, which is larger the further μ is from $r\mathbf{1}$ and smaller the larger the variances are.