Problem Set 2: Bellman Preliminaries and Covariance Matrices
FIN 539 Mathematical Finance
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Avantika and Jiaen will discuss this homework at the TA session, time TBA.

1. **Bellman Equation: preliminaries** This problem does some preliminary calculations for a problem we will solve in next week's homework.

Consider the HARA (Hyperbolic Absolute Risk Aversion) felicity (or utility) function $u(c) = (c - \underline{c})^{1-R}/(1-R)$, where $\underline{c}$ is the subsistence consumption (the minimal consumption needed to survive) and $R > 0$, $R \neq 1$, is the relative risk aversion for the increase of consumption above the subsistence level. Then we will study the following optimization problem:

Given w_0,
choose portfolio θ_t, consumption c_t, and wealth w_t to
maximize $E[\int_{t=0}^{\infty} e^{-\rho t} u(c_t) dt]$ (expected utility of lifetime consumption)
subject to:
$dw_t = rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t) - c_t dt$ (budget constraint)
$(\exists K \in \Re)(\forall t) w_t \geq -K$ (limited borrowing)

A. The Bellman equation is derived from dM_t for a process M_t defined in class which gives the realized value of the objective at time t given we are following a possibly suboptimal strategy until time t and then switching to the optimal strategy from then on. One thing we will have to compute in deriving dM_t is $d(e^{-\rho t}V(w_t))$, where $V(w_t)$ is the value function (as yet unknown, but assumed to be twice continuously differentiable) and dw_t is given by the budget constraint in the problem above. Use Itô's lemma to derive $d((e^{-\rho t}V(w_t))$.

Let $f(w_t, t) \equiv e^{-\rho t}V(w_t)$. Then

$$\begin{aligned} d((e^{-\rho t}V(w_t)) &= df(w_t, t) \\ &= f_w dw_t + f_t dt + \frac{1}{2} f_{ww} (dw_t)^2 \\ &= (rw_t dt + \theta_t((\mu - r)dt + \sigma dZ_t) - c_t dt) e^{-\rho t} V_w \end{aligned}$$

$$-\rho e^{-\rho t} V dt + \frac{\theta_t^2 \sigma^2}{2} e^{-\rho t} V_{ww} dt$$

B. Another term in deriving dM_t comes from taking a derivative of an integral with respect to parameters. This is ordinary calculus (Leibniz' rule), and the integral is done statewise. Compute $d(\int_{s=0}^t e^{-\rho s} u(c_s) ds)/dt$.

Only the upper limit of the integral depends on t, so the derivative is the derivative (=1) of the upper limit with respect to t times the value of the integrand at the upper limit. Therefore, we have

$$d(\int_{s=0}^t e^{-\rho s} u(c_s) ds) = e^{-\rho t} u(c_t) dt$$

C. Optimization of c at a point of time maximizes an objective function that equals $u(c) - cV_w$ (where $u(c) = (c - \underline{c})^{1-R}/(1-R)$) plus other terms that do not depend on c. Solve for the optimal c, and the maximized value of $u(c) - V_w c$. Note: V_w does not depend on c.

$$(c - \underline{c})^{-R} - V_w = 0$$

so

$$c^* = \underline{c} + (V_w)^{-1/R}.$$

At the optimum:

$$\begin{aligned} u(c^*) - V_w c^* &= \frac{((V_w)^{-1/R})^{1-R}}{1-R} - \underline{c} V_w - (V_w)^{-1/R} V_w \\ &= \frac{R(V_w)^{1-1/R}}{1-R} - \underline{c} V_w \end{aligned}$$

D. Optimization of θ at a point in time maximizes an objective function that equals $\theta(\mu - r)V_w + \theta^2\sigma^2 V_{ww}/2$. Solve for the optimal θ and the maximized value of $\theta(\mu - r)V_w + \theta^2\sigma^2 V_{ww}/2$. Note: V_w and V_{ww} do not depend on θ.

$$(\mu - r)V_w + \theta\sigma^2 V_{ww} = 0$$

so

$$\theta^* = -\frac{\mu - r}{\sigma^2}\frac{V_w}{V_{ww}}$$

At the optimum:

$$\begin{aligned}
\theta^*(\mu - r)V_w + (\theta^*)^2\sigma^2 V_{ww}/2 &= -\frac{\mu - r}{\sigma^2}\frac{V_w}{V_{ww}}(\mu - r)V_w \\
&\quad + \frac{(\mu - r)^2}{\sigma^4}\frac{(V_w)^2}{(V_{ww})^2}\frac{\sigma^2 V_{ww}}{2} \\
&= -\frac{(\mu - r)^2}{2\sigma^2}\frac{(V_w)^2}{V_{ww}}
\end{aligned}$$

2. **Positive definite covariance matrix** Suppose your client gives you the following 2×2 covariance matrix:

$$V = \begin{vmatrix} 0.0495 & 0.0505 \\ 0.0505 & 0.0495 \end{vmatrix}$$

(Okay, your client is more likely to give you a defective 10×10 covariance matrix, but I want this to be easy enough to solve by hand.)

A. Compute the eigenvalues of V. (Hint: to solve for the eigenvalues of A, use the equation $\det(A - \lambda I) = 0$.)

$$\det(V - \lambda I) = (.0495 - \lambda) * (.0495 - \lambda) - .0505 * .0505$$

$$\begin{aligned} &= \lambda^2 - .099\lambda + .0495^2 - .0505^2 \\ &= \lambda^2 - .099\lambda - .0001 \end{aligned}$$

By the quadratic formula,

$$\begin{aligned} \lambda &= \frac{.099 \pm \sqrt{.099^2 + 4 * .0001}}{2} \\ &= \frac{.099 \pm .101}{2} = .1 \text{ or } - .001 \end{aligned}$$

B. Show that V is not positive semi-definite.

Since V has a negative eigenvalue, it is not positive semi-definite.

C. Compute the normalized eigenvectors corresponding to the two eigenvalues. (Hint: use the equation $(A - \lambda_i I)x_i = 0$ to solve for the ith eigenvector of A.)

Denote the first eigenvector, corresponding to $\lambda_1 = .1$, as $x_1 = (x_{11}, x_{12})$. Then we have

$$0 = (V - .1I)x_1 = \begin{vmatrix} -0.0505 & 0.0505 \\ 0.0505 & -0.0505 \end{vmatrix} x_1,$$

so that $-.0505x_{11} + .0505x_{12} = 0$ and therefore $x_{12} = x_{11}$. Therefore, x_1 is proportional to $(1, 1)$ and we can write it as (x_{11}, x_{11}). To normalize it, $x_{11}^2 + x_{11}^2 = 1$, so we can take $x_{11} = 1/\sqrt{2}$ (choosing a factor $-1/\sqrt{2}$ would do just as well) and therefore $x_1 = (1, 1)/\sqrt{2}$.

Similarly, denote the first eigenvector, corresponding to $\lambda_1 = -.001$, as $x_2 = (x_{21}, x_{22})$. Then we have

$$0 = (V - (-.001)I)x_2 = \begin{vmatrix} 0.0505 & 0.0505 \\ 0.0505 & 0.0505 \end{vmatrix} x_2$$

so that $.0505x_{21} + .0505x_{22} = 0$ and therefore $x_{22} = -x_{21}$. Therefore, x_2 is proportional to $(1, -1)$ and we can write it as $(x_{21}, -x_{21})$. To normalize it, $x_{21}^2 + x_{21}^2 = 1$, so we can take $x_{21} = 1/\sqrt{2}$ (choosing a factor $-1/\sqrt{2}$ would do just as well) and $x_2 = (1, -1)/\sqrt{2}$.

D. Change any negative eigenvectors to 0.0001 and compute the new covariance matrix. (Hint: used the normalized eigenvectors and the formula $V = X'\Lambda X$.)

Let

$$X \equiv \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

and let

$$\hat{\Lambda} \equiv \begin{vmatrix} .1 & 0 \\ 0 & .0001 \end{vmatrix}$$

be the diagonal matrix with the new eigenvalues on the diagonal. Then the new covariance matrix is

$$\begin{aligned} \hat{V} &= X\hat{\Lambda}X' \\ &= \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \begin{vmatrix} .1 & 0 \\ 0 & .0001 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} 0.05005 & 0.04995 \\ 0.04995 & 0.05005 \end{vmatrix} \end{aligned}$$