Problem Set 1: Kuhn-Tucker conditions and positive semi-definiteness
FIN 539 Mathematical Finance
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Avantika and Jiaen will discuss this homework at the TA session, time TBA.

1. A mean-variance choice problem

Let the risky asset returns be given by a vector r that is normally distributed with mean vector μ and positive-definite covariance matrix V. You model a portfolio choice in a fund you manage using the following single-period optimization problem:

Choose risky portfolio proportions $\theta \in \Re^N$ to
maximize $E\left[-\frac{1}{A}\exp\left(-A\left(\theta' r - \gamma\frac{\mathrm{var}(\theta' r - \theta_B' r)}{2}\right)\right)\right]$
subject to:
$\mathbf{1}'\theta = 1$ (fully invested)

In this optimization problem, $A > 0$ is the absolute risk aversion, θ_B is the benchmark portfolio against which you are judged, and $\gamma \geq 0$ is the penalty for deviating from the benchmark. (Probably the penalty is soft and hard to quantify; hopefully this is a good proxy for the actual penalty.) We assume that θ_B is fully invested: $\mathbf{1}'\theta_B = 1$.

A. What are the choice variables? the objective function? equality constraints? inequality constraints?

The choice variables are the portfolio weights (θ's). The objective function is expected utility $E[...]$. Being fully invested ($\mathbf{1}'\theta = 1$) is the only equality constraint. There are no inequality constraints.

B. A normal random variable $x \sim N(m, \sigma^2)$ has a moment generating function $M(t) \equiv E[\exp(tx)] = \exp(tm + t^2\sigma^2/2)$. Use this formula and the formulas for mean and variance of a linear combination of random variables to rewrite the objective function without r and instead in terms of μ and V.

$E\left[-\frac{1}{A}\exp(-A\theta' r - \gamma\mathrm{var}(\theta' r - \theta_B' r)/2)\right]$

$$= -\frac{1}{A}\exp(-A\theta'\mu + A^2\frac{\theta' V\theta}{2} + A\gamma\frac{(\theta-\theta_B)' V(\theta-\theta_B)}{2})$$

C. Calculate the optimal portfolio as a function of the Lagrange multiplier on the constraint.

Let $K \equiv \exp(-A\theta'\mu + A^2\frac{\theta' V\theta}{2} + A\gamma\frac{(\theta-\theta_B)'V(\theta-\theta_B)}{2})$, evaluated at the optimal θ. Then, we can write the KT condition as

$$K(\mu - AV\theta - \gamma V(\theta - \theta_B)) = \lambda \mathbf{1},$$

where λ is the Lagrange multiplier for the only constraint. Since this is an equality constraint there is no restriction on the sign of λ or complementarity slackness condition.

We want to solve for θ. Bringing the terms with θ to the left-hand side, we have

$$(A+\gamma)V\theta = \mu + \gamma V\theta_B - \frac{\lambda}{K}\mathbf{1},$$

and therefore

$$\theta = \frac{A}{A+\gamma}\frac{V^{-1}\mu}{A} + \frac{\gamma}{A+\gamma}\theta_B - \frac{\lambda}{(A+\gamma)K}V^{-1}\mathbf{1}.$$

D. Use the fully invested constraint $\mathbf{1}'\theta = 1$ to solve for λ and substitute it into the expression for θ to obtain the solution in a form that doesn't depend on λ.

$$\begin{aligned} 1 &= \mathbf{1}'\theta \\ &= \frac{A}{A+\gamma}\frac{\mathbf{1}'V^{-1}\mu}{A} + \frac{\gamma}{A+\gamma}\mathbf{1}'\theta_B - \frac{\lambda}{(A+\gamma)K}\mathbf{1}'V^{-1}\mathbf{1}. \end{aligned}$$

Since $\mathbf{1}'\theta_B = 1$ and $1 - \gamma/(A+\gamma) = A/(A+\gamma)$, this implies that

$$-\frac{\lambda}{(A+\gamma)K} = \frac{A}{(A+\gamma)\mathbf{1}'V^{-1}\mathbf{1}}\left(1 - \frac{\mathbf{1}'V^{-1}\mu}{A}\right)$$

Therefore,

$$\theta = \frac{A}{A+\gamma}\frac{V^{-1}\mu}{A} + \frac{\gamma}{A+\gamma}\theta_B + \frac{A}{(A+\gamma)\mathbf{1}'V^{-1}\mathbf{1}}\left(1 - \frac{\mathbf{1}'V^{-1}\mu}{A}\right)V^{-1}\mathbf{1}.$$

E. The solution for the unconstrained problem is the same formula but with the Lagrange multiplier on the constraint equal to zero. Interpret how this solution changes as we vary the penalty γ on tracking error.

$$\theta = \frac{A}{A+\gamma}\frac{V^{-1}\mu}{A} + \frac{\gamma}{A+\gamma}\theta_B.$$

As γ increases from 0 to ∞, the solution starts from the solution $V^{-1}\mu/A$ that would be optimal absent a penalty for tracking error, and tends towards the benchmark θ_B as tracking error gets bigger and bigger. The weights on the two portfolios always add up to one, so the optimal solution is always a convex combination of the two.