INVESTMENTS
Lecture 3: supplement

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More on mean-variance optimization

- Single risky asset
- Adding a second uncorrelated asset

A Mean-Variance Optimization

Choose f to
maximize $r + f(\mu - r) + \lambda f^2 \sigma^2$

Variables and constants:

f fraction in the risky asset

r riskfree rate

$\mu > r$ mean return on the risky asset

$\sigma > 0$ standard deviation of return on the risky asset

$\lambda > 0$ risk aversion parameter

(advanced note) λ could be the Lagrange multiplier (shadow price) from a more general problem

A Mean-Variance Optimization: solution

$$\begin{aligned} objective &= r + f(\mu - r) + \lambda f^2 \sigma^2 \\ &= k - \lambda\sigma^2 \left(f - \frac{\mu - r}{2\lambda\sigma^2} \right)^2, \end{aligned}$$

where the exact value of the constant k is unimportant.[1] What is important is to note that the objective is maximized when

$$f = \frac{\mu - r}{2\lambda\sigma^2},$$

subtracting zero rather than a positive number. Note that the fraction invested is proportional to the excess return and inversely proportional to the variance. The proportion is smaller the more risk averse the investor.

[1] Actually, $k = r + (\mu - r)^2/4\lambda^2\sigma^2$.

Adding an uncorrelated risky asset

Choose f_1 and f_2 to
maximize $r + f_1(\mu_1 - r) + f_2(\mu_2 - r) + \lambda(f_1^2\sigma_1^2 + f_2^2\sigma_2^2)$

Variables and constants:

f_1 **and** f_2 fractions in the risky assets

r riskfree rate

$\mu_1 > r$ **and** $\mu_2 > r$ mean returns on the risky assets

$\sigma_1 > 0$ **and** $\sigma_2 > 0$ standard deviations of returns on the risky assets

$\lambda > 0$ risk aversion parameter

Note: the objective function is the sum of the objective functions we would have for each individual asset, so the fractions for the individual risky assets will be $f_i = (\mu_i - r)/2\lambda\sigma_i^2$. Again this is proportional to the excess return and inversely proportional to the variance.

Adding an uncorrelated risky asset: solution

$$\begin{aligned} objective &= r + f_1(\mu_1 - r) + f_2(\mu_2 - r) + \lambda(f_1^2\sigma_1^2 + f_2^2\sigma_2^2) \\ &= k - \lambda\left[\sigma_1^2\left(f_1 - \frac{\mu_1 - r}{2\lambda\sigma_1^2}\right)^2 + \sigma_2^2\left(f_2 - \frac{\mu_2 - r}{2\lambda\sigma_2^2}\right)^2\right], \end{aligned}$$

where the exact value of the constant k[2] is unimportant. What is important is to note that the objective is maximized when

$$f_1 = \frac{\mu_1 - r}{2\lambda\sigma_1^2}$$

and

$$f_2 = \frac{\mu_2 - r}{2\lambda\sigma_2^2},$$

subtracting zero rather than a positive number. As before, the fraction invested is proportional to the excess return and inversely proportional to the variance, and the constant of proportionality is inversely related to risk aversion.

[2]Actually, $k = r + (\mu_1 - r)^2/4\lambda^2\sigma_1^2 + (\mu_2 - r)^2/4\lambda^2\sigma_2^2$.