Problem Set 2: Optimization
FIN 550: Numerical Methods and Optimization in Finance
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Hand in your answers to Problem 2 (and optionally the answer to the "extra-for-experts" Problem 3) in class on Nov. 13. The answer to Problem 1 will be posted on the web site. If you choose to answer the challenger – Problem 4 – e-mail the answer to me.

1. Consider the following maximization problem:

Choose c_1 and c_2 to
maximize $3\log(c_1) + 2\log(c_2)$
subject to:
$c_1 + c_2 \leq 100$
and
$40 \leq c_2$

A. Write the constraints in the appropriate form for the Kuhn-Tucker conditions. Be careful to get the signs correct for the constraint functions.

B. Compute the gradient of the objective function and the gradient of each of the constraint functions.

C. Write down the Kuhn-Tucker conditions.

D. Solve the problem.

E. Prove that the solution is correct. (Probably, you will want to prove that a solution to the Kuhn-Tucker conditions with positive c_1, and c_2 satisfying the constraints of the problem must be an optimal solution.)

2. Consider a model of the stock market modeled using a complete market (Arrow-Debreu world) with four states of nature. The state probabilities are $\pi_1 = 0.1$, $\pi_2 = 0.4$, $\pi_3 = 0.4$, and $\pi_4 = 0.1$, and the corresponding state prices are $p_1 = 0.05$, $p_2 = 0.3$, $p_3 = 0.4$, and $p_4 = 0.15$. Assuming square root utility and initial wealth of $\$100,000$ and minimum consumption level of $\$100,000$, we have the following portfolio optimization problem:

Choose state-dependent consumption c_1, c_2, c_3, and c_4 to
maximize expected utility $\sum_{i=1}^{4} \pi_i \sqrt{c_i}$
subject to the budget constraint,
$\sum_{i=1}^{4} p_i c_i \leq 100000$
and the minimum consumption constraints,
$(\forall i) c_i \geq 100000$.

A. Write the constraints in the appropriate form for the Kuhn-Tucker conditions. Be careful to get the signs correct for the constraint functions.

B. Compute the gradient of the objective function and the gradient of each of the constraint functions.

C. Write down the Kuhn-Tucker conditions.

D. Solve the problem.

E. Prove that the solution is correct. (Probably, you want to prove that a solution to the Kuhn-Tucker conditions with positive c_1, c_2, c_3, and c_4 satisfying the constraints of the problem must be an optimal solution.)

3. (extra for experts – optional problem for students of superior ambition or background) Consider the following problem:[1]

Choose $\theta \in \Re$ to
maximize $\sin(\theta)$
subject to
$\tan(\Pi\theta) = 1$.

Show that this problem is feasible and bounded, but that there does not exist $\theta \in \Re$ that maximizes the objective function.

There is a hint for this problem at the following URL:

http://dybfin.wustl.edu/teaching/finopt12/homeworks/fo12hw2/hw2p3hint.pdf

[1] In this problem, Π is the constant ≈ 3.14159 that is the ratio of the circumference of a circle to its diameter, and the sine and tangent functions take arguments in radians.

4. (challenger – very tough problem, strictly an individual effort) Construct a $\mathcal{C}^\infty$ function $f : \Re \to \Re$ such that (a) $f(0) = 0$, (b) $(\forall x \neq 0) f(x) > 0$, and (c) all derivatives of f are zero at zero.