FIN 550 Exam Answers

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1. True-False (25 points)

A. Unconstrained problems typically have interior solutions.

True. (Unconstrained problems cannot have boundary solutions since there are no constraints.)

B. A local optimum of a linear program is a global optimum.

True.

C. Absence of arbitrage is an unusual feature for option pricing models.

False. Most option pricing models are based on the absence of arbitrage.

D. The product of the eigenvalues equals the determinant of a matrix.

True.

E. A linear program with an unbounded dual always has an optimal solution.

False. An unbounded dual implies the primal is infeasible and any optimal solution would have to be feasible.

2. Linear Programming (25 points)

Consider the following linear program:

Choose nonnegative x_1, x_2, and x_3 to
maximize $2x_1 + 2x_2 + 3x_3$, subject to
$x_1 + 2x_2 + 3x_3 \leq 6$ and
$x_1 + x_2 \leq 3$

An LP function in a program you have not used before gives the following solution to the LP.

choice variables: $x^* = (3, 0, 1)$

value of the program: 9

Lagrange multipliers of the constraints: $\lambda^* = (1, 1)$

Lagrange multipliers of positivity constraints: $\gamma^* = (0, 1, 0)$

A. Show that x^* is feasible. Obviously x^* satisfies nonnegativity. Also,

$$x_1^* + 2x_2^* + 3x_3^* = 6$$

and

$$x_1^* + x_2^* = 3$$

so both constraints are satisfied with equality.

B. Show that x^* is optimal. We confirm the first-order condition

$$\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

where the first two terms on the right are the Lagrange multiplier times the gradient for the regular constraints, and the final term is the Lagrange multiplier times the gradient for the positivity constraint with nonzero multiplier (and a negative sign because it is $\geq$ not $\leq$). Because LPs are convex optimization problems, the first-order conditions are sufficient (and necessary if the problem is non-degenerate).

The complementary slackness conditions are satisfied because the three nonzero multipliers correspond to constraints that hold with equality.

A nice alternative proof is to write down the dual problem and note that the claimed Lagrange multiplier vector $(1, 1)$ is feasible in the dual and has value 9 which is the same as the value of x^* in the primal. This implies that x^* is optimal (since no other feasible choice can have value bigger than the feasible choice $(1, 1)$ in the dual) and also that $(1, 1)$ is optimal in the dual. Note that this approach does not require us to solve the dual problem.

C. Confirm the value of the program.

$$2x_1^* + 2x_2^* + 3x_3^* = 2 \times 3 + 2 \times 0 + 3 \times 1 = 9$$

D. Use the Lagrange multipliers to approximate the new value of the program if we change 6 to 7 in the first constraint.

The change in value is approximately the change in the rhs times the Lagrange multiplier or $1 \times 1 = 1$, so the new value is approximately $9 + 1 = 10$. As a check on the direction of the change, relaxing the constraint should increase the value of the maximum, which it does.

3. REGIME-SWITCHING (25 points) Consider a two-state Markov Chain in continuous time. Regime switches take place at the following rates:

$$\left(\begin{array}{ll} \text{state } 1 \rightarrow \text{state } 2 & \text{probability } 0.05/\text{year} \\ \text{state } 2 \rightarrow \text{state } 1 & \text{probability } 0.05/\text{year} \end{array} \right)$$

Initially (at time $t = 0$), we are in state 1.

a. What is the matrix A in the ODE

$$\pi'(t) = A\pi(t)$$

describing the dynamics of the vector $\pi(t)$ of future regime probabilities?

$$A = \begin{pmatrix} -.05 & .05 \\ .05 & -.05 \end{pmatrix}$$

b. Solve for the eigenvalues of A.

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} -.05 - \lambda & .05 \\ .05 & -.05 - \lambda \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \lambda^2 + .1\lambda + .0025 - .0025 \\
&= \lambda(\lambda + .1)
\end{aligned}
$$

Therefore, $\lambda = 0, -.1$

c. Solve for the associated eigenvectors.

For $\lambda = 0$, $(A - \lambda I)x = 0$ implies:

$$\begin{pmatrix} -.05 & .05 \\ .05 & -.05 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Conjecture that $x_2 \neq 0$ and arbitrarily choose the scaling $x_2 = 1$. Then $-.05x_1 + .05x_2 = 0$ so $x_1 = x_2 = 1$. So taking this to be the first pair, it has eigenvalue $\lambda_1 = 0$ and eigenvector $x^1 = (1, 1)^T$.

For $\lambda = -.1$, $(A - \lambda I)x = 0$ implies:

$$\begin{pmatrix} .05 & .05 \\ .05 & .05 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Conjecture that $x_2 \neq 0$ and arbitrarily choose the scaling $x_2 = 1$. Then $.05x_1 + .05x_2 = 0$ so $x_1 = -x_2 = -1$. This is the second pair, with eigenvalue $\lambda_2 = -.1$ and eigenvector $x^1 = (1, -1)^T$.

To check our work, confirm that

$$Ax^1 = \begin{pmatrix} -.05 & .05 \\ .05 & -.05 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$Ax^2 = \begin{pmatrix} -.05 & .05 \\ .05 & -.05 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} .1 \\ -.1 \end{pmatrix} = -.1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

d. Write down the general solution of the ODE.

$$
\begin{aligned}
\pi(t) &= c_1 e^{\lambda_1 t} x^1 + c_2 e^{\lambda_2 t} x^2 \\
&= c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-.1t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\end{aligned}
$$

e. Write down the particular solution corresponding to the initial condition that we start in state 1.

So $\pi(0)$ is $(1,0)^T$ and must equal the general solution at $t=0$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

It is easy to solve for $c_1 = 1/2$ and $c_2 = -1/2$ so

$$\begin{aligned} \pi(t) &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1+\exp(-.1t) \\ 1-\exp(-.1t) \end{pmatrix} \end{aligned}$$

f. A project costing \$110,000 has a cash flow of \$12,000/year in state 1 and \$6,000/year in state 2. The cash flows continue forever. If the continuously-compounded interest rate is 10%/year, does this project have a positive NPV?

In thousands:

$$\begin{aligned} PV &= \int_{t=0}^{\infty} \begin{pmatrix} 12 \\ 6 \end{pmatrix}^T \pi(t) e^{-.1t} dt \\ &= \int_{t=0}^{\infty} (12 \times \frac{1}{2}(1+e^{-.1t}) + 6 \times \frac{1}{2}(1-e^{-.1t}))e^{-.1t} dt \\ &= \int_{t=0}^{\infty} 9e^{-.1t} dt + \int_{t=0}^{\infty} 3e^{-.2t} dt \\ &= \frac{9}{.1} + \frac{3}{.2} \\ &= 105 \end{aligned}$$

Therefore, the net present value is $-110,000 + 105,000 = -5,000$. The project does not have a positive NPV.

4. Kuhn-Tucker Conditions (25 points)

Consider the following optimization problem:

Choose c_u and c_d to
maximize $\frac{1}{2}(20c_u - c_u^2) + \frac{1}{2}(20c_d - c_d^2)$, subject to
$\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) = 6$.

This is a single-period choice of investment for consumption in a binomial model with quadratic utility, initial wealth of 6, actual probabilities 1/2 and 1/2, risk-neutral probabilities 1/4 and 3/4, and riskfree rate of 25% (and therefore discount factor 4/5).

A. What are the objective function, choice variables, and constraint?

The objective function is $\frac{1}{2}(20c_u - c_u^2) + \frac{1}{2}(20c_d - c_d^2)$, the choice variables are c_u and c_d, and the constraint is $\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) = 6$ or equivalently $\frac{1}{5}c_u + \frac{3}{5}c_d = 6$.

B. What are the Kuhn-Tucker conditions?

$$\begin{pmatrix} 10 - c_u \\ 10 - c_d \end{pmatrix} = \lambda \begin{pmatrix} 1/5 \\ 3/5 \end{pmatrix}$$

Since the constraint is an equality constraint, there are not any complementary slackness conditions. (Sometimes we include in the Kuhn-Tucker condition the constraint from the original problem, $\frac{4}{5}(\frac{1}{4}c_u + \frac{3}{4}c_d) = 6$.)

C. If we add constraints $c_u \geq 6$ and $c_d \geq 6$, what are the Kuhn-Tucker conditions now?

$$\begin{pmatrix} 10 - c_u \\ 10 - c_d \end{pmatrix} = \lambda \begin{pmatrix} 1/5 \\ 3/5 \end{pmatrix} + \gamma_u \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \gamma_d \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

with $\gamma_u \geq 0$, $\gamma_d \geq 0$, and the complementary slackness conditions $\gamma_u(6 - c_u) = 0$ and $\gamma_d(6 - c_d) = 0$. The minus signs in the γ_u and γ_d terms arise because we are converting $\geq$ to $\leq$. (Sometimes we include in the Kuhn-Tucker condition the constraints from the original problem, which are $\frac{4}{5}(\frac{1}{4}c_u+\frac{3}{4}c_d) = 6$, $6-c_u \leq 0$, and $6 - c_d \leq 0$.)

5. Bonus question (30 bonus points)

A. Solve the optimization problem in problem 4 without the extra constraints in part 4C.

From the first-order condition in part 4B, we have that $c_u = 10 - \lambda/5$ and $c_d = 10 - 3\lambda/5$. Plugging into the budget constraint $c_u/5 + 3c_d/5 = 6$ and solving for λ, we obtain $\lambda = 5$ and therefore $c_u = 9$ and $c_d = 7$. Since the constraint set is convex (because it is the intersection of linear equalities and inequalities) and the objective function is strictly concave (verified shortly), this is a unique optimum. To verify that the objective function is strictly concave, notice that the matrix of second partials is

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is negative definite because it has all negative eigenvalues -1 and -1. We know this because either (1) we know that the eigenvalues of a diagonal matrix are its diagonal elements, or (2) we write down the characteristic equation $0 = \det(M - \lambda I) = (1 + \lambda)^2$.

B. Solve the optimization problem in problem 4 with the extra constraints in part 4C.

The unconstrained solution satisfies the constraints, so choosing $c_u = 9$ and $c_d = 7$ is still the solution.