Problem Set 4: Eigenvalues, eigenvectors, and regime-switching models
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Answers to all these problems (except Extra for Experts) are on the course page. You don't have to submit any answers because I don't want to require you to hand in an assignment during the reading period. However, I strongly recommend working the problems and checking your answers because there is sure to be a Markov switching problem on the final.

1. Consider the matrix

$$D = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

A. Compute the eigenvalues λ_1 and λ_2 of A.

B. Compute corresponding eigenvectors.

C. Let $x_0 = (3, 2)^T$. Write x_0 as a linear combination of the eigenvectors.

D. Use the eigenvalues and eigenvectors to compute $A^5 x_0$.

2. Extra for Experts (an optional problem for students of superior ability, preparation, or ambition) Prove your answers. There is no need to check the answer using matlab (which would be a challenge in itself).

Let

$$F = \begin{pmatrix} 3 & 7 \\ 2 & 8 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) Then, how many digits are there in the largest element of

$$y = F^{100000} g?$$

(b) What is the difference between the first element of y and the second element of y?

3. Consider a model with three economic scenarios: (1) healthy economy, (2) recession, and (3) depression. These states are assumed to follow a Markov switching model in continuous time. From a healthy economy, the economy has a probability per unit time of .05 of moving to a recession but cannot move

directly to a depression. From a recession, the economy has a probability per unit time of .03 of moving to a healthy economy and a probability per unit time of .02 of moving to a depression. From a depression, the economy has a probability per unit time of .05 of moving to a recession but cannot move directly to a healthy economy.

A. Let $\pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))^T$ be the vector of the probabilities of the three states at a future time t given the information now. Write down a first-order vector ODE satisfied by $\pi(t)$.

B. Find the general solution of the vector ODE given in part A.

C. Find the solution of the ODE that satisfies the initial condition that we are in a recession at time $t = 0$.

D. We have a possible investment project that requires an initial investment of $\$100,000$. The project pays a cash flow c_t of $\$7,000$/year when the economy is healthy, $\$1,000$/year in a recession, and $\$0$/year in a depression. If the interest rate is 2%, is the net present value

$$\int_{t=0}^{\infty} e^{-rt} E[c_t] dt - 100,000$$

of the cash flows positive?

E. (extra for experts) Suppose this investment does not have to be made immediately and instead we can wait and undertake the investment at the first time τ the economy is healthy. Then what is the net present value

$$E\left[\int_{t=\tau}^{\infty} e^{-rt} c_t dt - 100,000 e^{-r\tau}\right]$$

for this strategy? Is this more or less valuable than investing immediately?