DERIVATIVE SECURITIES
Lecture 5: Fixed-income securities

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- Interest rates
- Interest rate derivative pricing: general issues
- Bond and bond option pricing
- Interest rate swaps
- Caps, floors, and collars

Binomial asset pricing: review

Riskless bond: $\quad 1 \quad \longrightarrow \quad r$

Stock: $\quad 1 \quad < \begin{array}{l} u \\ d \end{array}$

Derivative security: $\quad ??? \quad < \begin{array}{l} V_u \\ V_d \end{array}$

$$Value = p_u V_u + p_d V_d = r^{-1}(\pi_u^* V_u + \pi_d^* V_d)$$

where

$$p_u = r^{-1}\frac{r-d}{u-d} \quad p_d = r^{-1}\frac{u-r}{u-d}$$

and

$$\pi_u^* = \frac{r-d}{u-d} \quad \pi_d^* = \frac{u-r}{u-d}.$$

Interpretation: all investments are fair gambles, subject to discounting to account for impatience and a probability adjustment to account for risk pricing.

“The” interest rate

What is “the” interest rate?

- spot riskfree rate
- discount interest rate (different maturities)
- forward rate (different maturities)
- coupon bond rates
- LIBOR/SHIBOR etc.
- corporate bond rates
- municipal bond rates
- mortgage rates

For this class, we will focus on “riskless” (default-free) bonds. They still have interest rate risk depending on the maturity.

Risk-neutral probabilities: the interest rate is not an asset

The interest rate is not an asset *or* a futures. For option pricing, it is most convenient to work with the short (spot) interest rate. We could take the short interest rate to follow a binomial process of some sort:

$$r_0 \quad \begin{matrix} \nearrow & r_1^U \\ \searrow & r_1^D \end{matrix}$$

Then there is a riskless asset in which we can invest 1 today and get $1 + r_0$ tomorrow:

$$1 \quad \longrightarrow \quad 1 + r_0$$

but it would be unreasonable to expect there is an asset (or futures contract) with price r_0 today and a price of either r_1^U or r_1^D tomorrow. Therefore, we don't have an arbitrage with two assets to determine the risk-neutral probabilities.

Note on timing: we learn the riskfree rate at the start of the period, and we index by the time we learn it. Instead, we could index by the time the return is realized (as we usually do for stock returns).

Risk-neutral probabilities: work off some multiperiod bond

Riskless bond:

$$1 \quad \longrightarrow \quad 1 + r_0$$

Long bond

$$B \quad < \begin{matrix} UB \\ DB \end{matrix}$$

Derivative security:

$$??? \quad < \begin{matrix} V_U \\ V_D \end{matrix}$$

awkward: what bond to use? how to decide on U and D?

Risk-neutral probabilities: practical approaches

There are two usual solutions in practice:

(1) Set the risk-neutral probabilities equal to our estimate for the actual probabilities. Economic theory does not give us a good idea of even the sign of the risk premium, so this is reasonable.

(2) Back out the mean change in interest rates in the risk-neutral probabilities from the yield curve.

Often, there is a sort of combination of these two approaches...

Multi-period valuation

When we are working with risk-neutral valuation in a model with non-trivial interest rates, we have to discount by the rolled-over short interest rate. Absent dividends, we value an asset by:

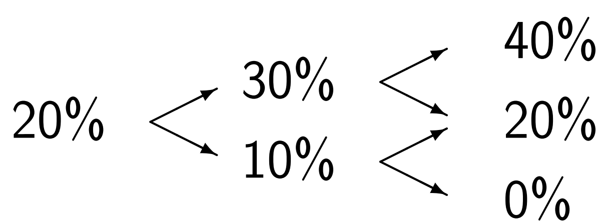
$$\begin{aligned} V_0 &= E^*[\frac{1}{(1+r_0)}V_1] \\ &= E^*[\frac{1}{(1+r_0)}\frac{1}{(1+r_1)}V_2] \\ &= E^*[\frac{1}{(1+r_0)}\frac{1}{(1+r_1)}\frac{1}{(1+r_2)}\dots\frac{1}{(1+r_{T-1})}V_T] \\ &= E^*[\frac{1}{(1+r_0)(1+r_1)(1+r_2)\dots(1+r_{T-1})}V_T] \end{aligned}$$

where V_t is the value of the asset at time t and r_t is the riskfree interest rate quoted at time t for an investment to time $t+1$. To use the binomial model, we discount back through the tree one node at a time (as usual), being careful to use the correct interest rate at each node. To use simulations, we take the sample mean of many draws from the random present value.

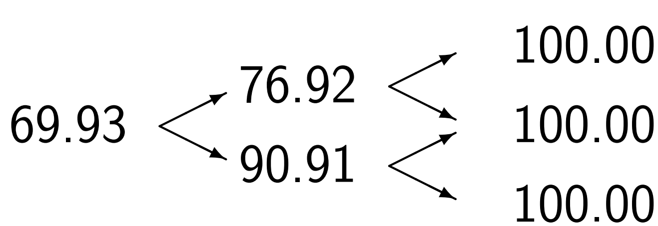
Example: valuation of a riskless bond

Consider a two-period binomial model in which the short riskless interest rate starts at 20% and moves up or down by 10% each period (i.e., up to 30% or down to 10% at the first change). The artificial probability of each of the two states at any node is $1/2$. What is the price at each node of a discount bond with face value of $\$100$ maturing two periods from the start?

The interest tree is



We can compute the bond price tree from the end and solving back:



In-class exercise: bond and bond option valuation

Consider a two-year binomial model. The short riskless interest rate starts at 50% and moves up or down by 25% each year (i.e., up to 75% or down to 25% at the first change). The artificial probability of each of the two states at any node is $1/2$. What is the price at each node of a discount bond with face value of $\$700$ maturing two periods from the start?

What is the value at each node of an American call option on this discount bond with a strike price of $\$500$ and maturity one year from now?

Interest rate swap

There are lots of different kinds of swaps. One of the simplest swaps a floating rate for a fixed rate. This can be used by one of the parties to convert a fixed-rate loan into a variable-rate loan or vice versa. Converting borrowing in the spot market (which is like a variable-rate loan) into a fixed-rate loan can be done by buying a swap of floating into fixed, and would be attractive for a company whose management thinks the credit will improve more over time than is implicit in the rates at which they can do term borrowing. In this case, the economic function of the swap from the company's perspective is to separate the company's own credit risk (which the company wants to bear) from the general interest rate risk in the market.

In this class, we are not going to consider credit or counterparty risk, and instead we will look at a pure swap of the floating riskless rate for a fixed riskless rate that doesn't change over time. This is a good benchmark for a company that does face counterparty default risk, and it may be a very good approximation to the value if the counterparty has very good credit or very good collateral.

Interest rate swap: theoretical value of the swap rate

Consider an interest rate swap that, in each period, has an exchange of the floating rate r_t for the constant rate s. What is the swap rate s that makes this fairly priced without cash changing hands at the beginning?

If we knew s, we could do valuation for future payments $s - r_t$ at each time t, and then solve for s by trial-and-error. However, we can make use of the linearity of option pricing to do this in one step by looking at the value of getting s in each period (which will be just a constant times s) and setting that equal to the value that makes it equal to the value of getting r_t at each future time t.

For both legs of the swap, the valuation is the same as for any asset that is throwing off cash flows: the value at a node is the continuation value plus the value of any cash flow thrown off by the asset. We have to be careful to use a consistent convention for whether the value at a node is computed before or after the cash flow. The swap itself is the difference between two assets (the fixed leg and the floating leg) so it satisfies the same equation as an asset: even though it starts out with value zero like a futures contract, the value is not reset to zero every day.

Interest rate swap: example

Consider a two-period binomial model in which the short riskless interest rate starts at 20% and moves up or down by 10% each period (i.e., up to 30% or down to 10% at the first change). The artificial probability of each of the two states at any node is $1/2$. What is the swap rate for a fixed-for-floating swap with payments at the next two points of time?

The interest tree is

		40%
	30%	
20%		20%
	10%	
		0%

floating leg (pre-cash flow values)

		40
	53.1	
30.07		20
	19.1	
		0

fixed leg (pre-cash flow values)

		s
	1.77s	
1.53s		s
	1.91s	
		s

$$30.07 = 1.53s \quad \Rightarrow \quad s = 19.6\%$$

Interest rate swap: units, swaptions

The swap rate is usually expressed as an annual percentage, and swaps usually pay periodically at intervals smaller than a year. There is a *notional amount* of the loan underlying the swap, and the swap payments are computed on that basis. For example, if we buy a fixed-for-floating swap that pays every three months with notional amount $\$1,000,000$, if the swap rate is 3% and the floating rate 1%, we will receive $(3\% - 1\%) \times (3/12) \times \$1,000,000 = \$5,000$ this quarter. Usually there are some more details we need to be concerned with (e.g. exactly what interest rate do we use for the floating side, at what date is the rate determined and when is the cash flow due).

A swaption is an option to enter a swap at a prespecified swap rate. Valuation works the same as valuing other options once we have computed the swap rate at different nodes in the tree.

Caps and floors

If you are borrowing at a floating rate and you want to limit the amount of interest you pay, you can buy an interest rate cap. A cap is a portfolio of call options on the interest rate, options maturing periodically between now and maturity, paying the excess, if any, of the floating interest over the cap rate. This implies that the interest payment less the payout from the cap is never more than the cap rate.

If you are lending at a floating rate and you want to guarantee a minimum amount of interest you will receive, you can buy an interest rate flor. A floor is a portfolio of put options on the interest rate, options maturing periodically between now and maturity, paying the excess, if any, of the floor rate above the floating interest. This means that for a lender, the interest from the floating-rate loan plus the floor payoff is never less than the floor rate.

Like swaps, caps and floors are based on notional amounts and typically quoted using annual interest rates although the payments usually cover shorter terms.

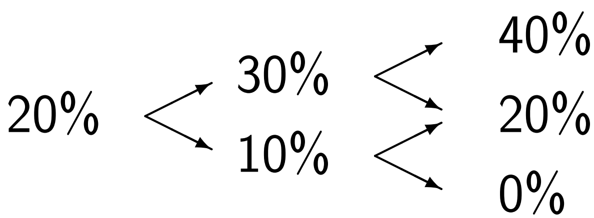
Collar = cap - floor

If you want to buy a cap but you don't want to pay for it up-front, you can buy a collar instead, which is long a cap and short a floor. You use the floor to pay for the cap (at least in part): if interest rates go up your interest expense is limited, but on the other hand your savings are limited if interest rates go down (which is how you pay fot the cap).

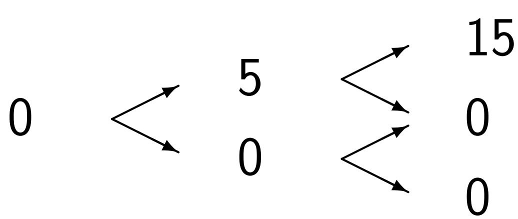
Cap example

Consider a two-period binomial model in which the short riskless interest rate starts at 20% and moves up or down by 10% each period (i.e., up to 30% or down to 10% at the first change). The artificial probability of each of the two states at any node is $1/2$. What is the price of a cap at 25% with payments at the next two points of time?

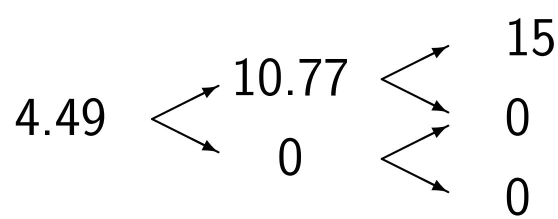
The interest tree is



cap cash flows



cap price (pre-cash flow values)



In-class exercise: collar

Consider a two-year binomial model. The short riskless interest rate starts at 50% and moves up or down by 25% each year (i.e., up to 75% or down to 25% at the first change). The artificial probability of each of the two states at any node is $1/2$. What is the price of a collar with a cap of 58% and a floor of 30%?

Additional interesting topics (a few of millions!)

- FX options
- Bond futures and the delivery option
- Corporate bonds and pricing of default risk
- Credit Default Swaps
- Stochastic Volatility (CIR, Heston, ARCH, GARCH, SV, etc.)