DERIVATIVE SECURITIES
Lecture 1: Background, Arbitrage, and Binomial Model

Philip H. Dybvig
Washington University in Saint Louis

- applications
- derivatives market players
- big ideas
- strategy example
- single-period binomial model
- artificial (risk-neutral) probabilities
- multiple-period binomial model

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Derivatives pricing theory is useful for...

- trading marketed options and futures
- pricing corporate claims like convertibles and warrants
- pricing or choosing when to exercise employee stock options (ESOs)
- valuing contract terms (e.g. an option to renew a lease)
- choosing a mortgage (evaluating the refinancing option)
- risk management
- designing dynamic investment strategies
- capital budgeting taking into account decisions in future contingencies
- extracting market expectations from quoted prices (e.g. implied volatilities)
- speculating on economic events
- ...and more

Some definitions

- derivative security ... a contract whose value derives from (depends on) something else
- underlying ... what the value depends on, for example,
  - a commodity price (e.g. oil, gold, wheat, orange juice, electricity)
  - an exchange rate (e.g. yen-dollar, euro-yen)
  - a security price (e.g. IBM common stock, Treasury Bond)
  - credit
  - volatility (vol)
  - a stock index (e.g. S&P 500)
  - other economic factors (e.g. inflation or GDP)
- arbitrage ("arb") ... a strategy generating profit without any downside, a money pump
- volatility ("vol") ... variance of an underlying

Some reasons for derivative trades

- hedging ... using derivatives to offset risk (insurance)
- risk management ... evaluation and management of risks
- speculation ... taking on a derivative position to bet on market moves
- market manipulation ... trying to move the market to create a profit (warning: probably illegal) opportunity
- market making ... trading to profit from providing liquidity
- liquidity trade ... a trade not based on useful proprietary information, often a sale to generate cash
- informed trade ... a trade based on useful proprietary information

Some examples of derivatives

- forward contract ... a commitment to buy at a future date at a fixed price
- futures contract ... like a forward contract, but marked to market daily
- call option ... the buyer of the option has the right, but not the obligation, to buy in the future at a fixed price
- put option ... the buyer of the option has the right, but not the obligation, to sell in the future at a fixed price
- European option ... option can be exercised at maturity only
- American option ... option can be exercised anytime before maturity
- lookback option ... option based on price path, e.g. average price over time
- up-and-out (resp. down-and-out) option ... option that disappears when the underlying hits an upper (resp. a lower) boundary.
- up-and-in (resp. down-and-in) option ... option that becomes active when the underlying hits an upper (resp. a lower) boundary.
- futures or option on stock volatility, e.g. VIX

More examples of derivatives

- Swap ... agreement to swap one cash flow stream for another
- Swaption ... option to enter a swap at a fixed price in the future
- ESOP ... Employee Stock Option Plan
- real option ... option embedded in the operations of a firm (many examples)
- tax timing option ... real option motivated by the tax code
- reload option ... a particular type of employee stock option
- call provision ... option to pay off a corporate bond early
- pension liability ... what is owed to participants in an pension plan
- collateralized mortgage obligation ... claim to part or all of the cash flows thrown off by a pool of mortgages
- credit derivative ... option providing insurance against a specified list of possible default events, e.g. credit default swap (CDS)
- convertible bond ... corporate bond that can be converted to equity at the option of the buyer

Some big ideas

- More is better than less.
- Buy low.
- Sell high.
- Hedge and borrow or lend to convert a profit opportunity into an arb.
- Absence of arbitrage is usually a good approximation.
- Be skeptical if you think you found an arb.

Story about a profit opportunity (hedging example)

High grade copper trades today in the spot market at \$1.30 a pound. We can make a private deal to buy 100,000 pounds of scrap copper in the spot market for \$90,000 (\$0.90 per pound), and at a cost of \$0.30 a pound today (including transportation and all other costs) a smelter will convert the scrap to high-grade copper to be delivered six months from now.

A naive calculation makes this seem like a good deal: \$1.30 - 0.90 -0.30 = \$0.10. However, how do we make sure this is a good deal? How do we structure the deal so we do not take on a huge amount of risk?

Cash flows from buying and smelting the copper

time	now	six months out
purchase	(\$90K)	-
scrap	scrap Cu +100K lbs.	
refining	scrap Cu (100K) lbs.	high grade Cu +100K lbs.
copper	(\$30K)	-
sell refined	-	high grade Cu (100K) lbs.
copper spot	-	unknown price/lb $\times$ +100K
net	(\$120K)	unknown price/lb $\times$ +100K

Looking at this, it is not so obvious that this is a good deal. We don't know what the price of copper will be six months from now. We could probably build some sophisticated model predicting what the price will be, but that would still leave us with the possibility of a big loss if prices fall. Also, we are not experts on the copper market and we may miss something important that is known to sophisticated participants. Derivatives give us a better path.

Hedging

Selling copper forward (instead of selling later in the spot market) is a way of removing the price risk. A forward contract commits to trade in the future. The date ("maturity"), quantity, and price ("forward price") are constants dictated by the forward contract. In our example, suppose we find out that the 6-month forward price of copper is \$1.25/pound. This is disappointing; market participants expect that prices will fall over the next six months. Using the forward contract to eliminate the price risk we have:

time	now	six months out
purchase	(\$90K)	-
scrap	scrap Cu +100K lbs.	
refining	scrap Cu (100K) lbs.	high grade Cu +100K lbs.
copper	(\$30K)	-
sell refined	-	high grade Cu (100K) lbs.
copper forward	-	\$125K
net	(\$120K)	\$125K

Financing

We need cash to buy the scrap and pay for refining: proceeds from selling the refined copper will only come in six months time. We shop for a loan and find best offer against the copper has a 5% rate (straight interest), implying a rate of $5\% \times 6 \div 12 = 2.5\%$ over 6 months. With borrowing we have

time	now	six months out
purchase	(\$90K)	-
scrap	scrap Cu +100K	
refining	scrap Cu (100K) lbs.	high grade Cu +100K lbs.
copper	(\$30K)	-
sell refined	-	high grade Cu (100K) lbs.
copper forward	-	\$125K
borrowing	\$120K	(\$123K)
net	-	\$2K

The hedging and financing have converted the profit opportunity into an arb (arbitrage). The next step is skepticism: have we omitted any legal costs or fees? Is our counterparty for the forward contract a good credit? Is \$2K enough to compensate us for our time?

Why thinking about arbitrage is crucial

- key to making money
- key to valuation

cautions:

- absence of arbitrage is the norm.
- most arbs still involve some risk-taking.
- many smart people are looking for arbs; finding simple arbs is especially rare.

In-class exercise

Suppose that the spot price for high-grade copper is \$1.25/pound, that the 6-month forward price for high-grade copper is \$1.35/pound, and that the cost of storing copper for 6 months is \$0.03/pound, payable in advance. Assume that we can borrow at straight interest of 5%/year (=2.5% over 6 months). Set up the cash flows from a candidate arbitrage from storing copper at a scale of 100,000 pounds. Is it a good deal?

time	now	six months out
buy Cu spot		
store Cu for 6 months		
sell Cu forward		
borrow		
net		

Wrap-up (so far)

Big ideas!

- More is better than less.
- Buy low. (Buy if it is cheap.)
- Sell high. (Sell if it is expensive.)
- Hedge and borrow or lend to convert a profit opportunity into an arb.
- Absence of arbitrage is usually a good approximation.
- Be skeptical if you think you found an arb.

The binomial option pricing model

The option pricing model of Black and Scholes revolutionized a literature previously characterized by clever but unreliable rules of thumb. The Black-Scholes model uses continuous-time stochastic process methods that interfere with understanding the simple intuition underlying these models. We will start instead with the binomial option pricing model of Cox, Ross, and Rubinstein, which captures all of the economics of the continuous time model but is simple to understand and program. For option pricing problems not appropriately handled by Black-Scholes, some variant of the binomial model is the usual choice of practitioners since it is relatively easy to program, fast, and flexible.

Cox, John C., Stephen A. Ross, and Mark Rubinstein (1979) "Option Pricing: A Simplified Approach" *Journal of Financial Economics* **7**, 229–263

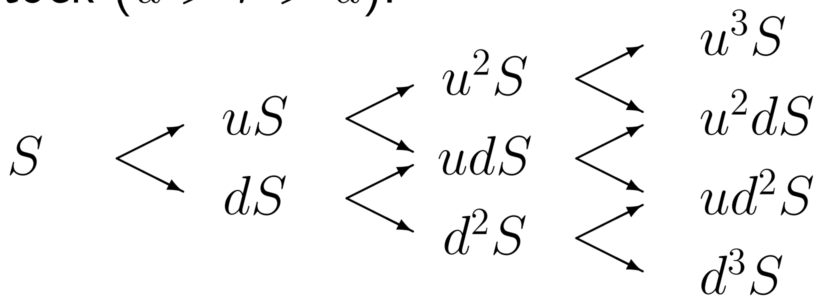
Black, Fischer, and Myron Scholes (1973) "The Pricing of Options and Corporate Liabilities" *Journal of Political Economy* **81**, 637–654

Binomial process (3 periods)

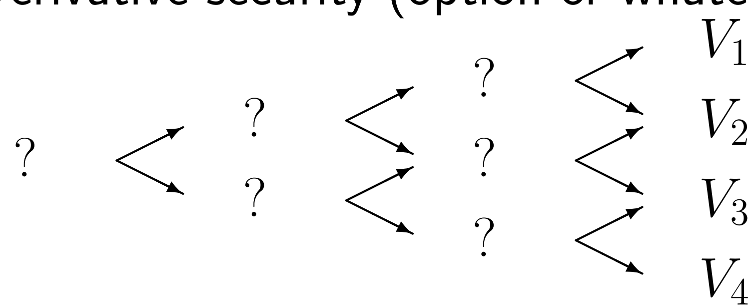
Riskless bond:

$$1 \longrightarrow r \longrightarrow r^2 \longrightarrow r^3$$

Stock ($u > r > d$):



Derivative security (option or whatever):



What is the price of the derivative security?

A simple option pricing problem in one period

Riskless bond (interest rate is 0): $100 \longrightarrow 100$

Stock:

$$50 \quad \begin{array}{l} \nearrow \; 100 \\ \searrow \; 25 \end{array}$$

Derivative security (at-the-money call option):

$$? \quad \begin{array}{l} \nearrow \; 50 \\ \searrow \; 0 \end{array}$$

To duplicate the call option with α_S shares of stock and α_B bonds:

$$\begin{aligned} 50 &= 100\alpha_S + 100\alpha_B \\ 0 &= 25\alpha_S + 100\alpha_B \end{aligned}$$

Therefore $\alpha_S = 2/3$, $\alpha_B = -1/6$, and the duplicating portfolio is worth $50\alpha_S + 100\alpha_B = 100/6 = 16\ 2/3$. By absence of arbitrage, this must also be the price of the call option.

In-class exercise: one-period contingent claim valuation

Compute the duplicating portfolio and the price of the general derivative security below. Assume $u > r > d > 0$.

Riskless bond:

$$1 \longrightarrow r$$

Stock:

$$1 \;\begin{array}{l} \nearrow \; u \\ \searrow \; d \end{array}$$

Derivative security:

$$??? \;\begin{array}{l} \nearrow \; V_u \\ \searrow \; V_d \end{array}$$

Multi-period valuation and artificial probabilities

In general, exactly the same valuation procedure is used many times, taking as given the value at maturity and solving back one period of time until the beginning. This valuation can be viewed in terms of state prices p_u and p_d or risk-neutral probabilities π_u and π_d, which give the same answer (which is the only answer consistent with no arbitrage):

$$Value = p_u V_u + p_d V_d = r^{-1}(\pi_u V_u + \pi_d V_d)$$

where

$$p_u = r^{-1}\frac{r-d}{u-d} \quad p_d = r^{-1}\frac{u-r}{u-d}$$

and

$$\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}.$$

Interpretation: all investments are fair gambles, subject to discounting to account for impatience and a probability adjustment to account for risk pricing.

In-class exercise: digital option

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 1$. What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of a *digital option* that pays \$100 when the final stock price is greater than \$120 and pays \$0 otherwise.

stock

digital option

hint: Start by filling in the stock value tree. Then compute the option values at the end. Finally, use single-period valuation to step back through the tree one node at a time.
reminder: $\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}$.