Practice problems for Lecture 4: answers

1. Black-Scholes option pricing

Suppose the stock price is 40 and we need to price a call option with a strike of 45 maturing in 4 months. The stock is not expected to pay dividends. The continuously-compounded riskfree rate is 3%/year, the mean return on the stock is 7%/year, and the standard deviation of the stock return is 40%/year. (The Black-Scholes formula is given at the end of the homework.)

a. What are S and B?

$S = 40$ (the current stock price as given), and $B = 45\exp(-.033(1/3)) \approx 44.55224$. Note that $4/12 = 1/3$ is the time to maturity.

b. What are x_1 and x_2?

$$\begin{aligned} x_1 &= \frac{\log(S/B)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \\ &= \frac{\log(40/44.55224)}{.4\sqrt{1/3}} + \frac{1}{2}.4\sqrt{1/3} \\ &= -0.3512442 \end{aligned}$$

$$\begin{aligned} x_2 &= \frac{\log(S/B)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \\ &= \frac{\log(40/44.55224)}{.4\sqrt{1/3}} - \frac{1}{2}.4\sqrt{1/3} \\ &= -0.5821843 \end{aligned}$$

c. $N(x_1) = 0.3627026$ and $N(x_2) = 0.2802213$ (confirm for yourself if you like). What is the Black-Scholes call price?

$$\begin{aligned} C &= SN(x_1) - BN(x_2) \\ &\approx 40 \times 0.3627026 - 44.55224 \times 0.2802213 \\ &\approx 2.023617 \end{aligned}$$

d. What is the Black-Scholes price for the European put with the same strike and maturity?

By put-call parity, $S + P = B + C$, and therefore the price is

$$\begin{aligned} P &= B + C - S \\ &\approx 44.55224 + 2.023617 - 40 \\ &\approx 6.57586 \end{aligned}$$

e. Conceptual question: Since the put option is worth more alive than if exercised now, can we conclude that an American version of the put is worth the same as the European put?

No. The value today is enhanced by the option to exercise an American option anytime between now and maturity. The value is usually higher because of the value of exercising later but before maturity in some contingencies.

2. Approximation

As noted in class, for near-the-money call options, a good approximation of the option price near maturity is

$$C \approx \frac{S-B}{2} + .4\frac{S+B}{2}\sigma\sqrt{T}$$

where S is the stock price, B ("the bond price") is the present value of receiving the strike at maturity, σ is the local standard deviation, and T is the time to maturity.

Consider an at-the-money call option that is one week to maturity on a stock with a price of \$50/share and a local standard deviation of 35%/year. The continuously-compounded riskfree rate is 1%/year.

a. What is the call price from the approximate formula?

$T = 1/52$, $S = 50$, $B = 50exp(-.1 \times 1/52) \approx 49.9903855$, $\sigma = .35$

$$\begin{aligned}
C &\approx \frac{S-B}{2} + .4\frac{S+B}{2}\sigma\sqrt{T} \\
&\approx \frac{50 - 49.9903855}{2} + .4\frac{50 + 49.9903855}{2}.35\sqrt{\frac{1}{52}} \\
&\approx .9754392
\end{aligned}$$

b. What is the call price from Black-Scholes?

$$\begin{aligned}
x_1 &= \frac{\log(S/B)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \\
&\approx \frac{\log(50/49.9903855)}{.35\sqrt{1/52}} + \frac{1}{2}.35\sqrt{1/52} \\
&\approx 0.02823029
\end{aligned}$$

$$\begin{aligned}
x_2 &= \frac{\log(S/B)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T} \\
&\approx \frac{\log(50/49.9903855)}{.35\sqrt{1/52}} - \frac{1}{2}.35\sqrt{1/52} \\
&\approx -0.02030597
\end{aligned}$$

$$\begin{aligned}
C_{BS} &= SN(x_1) - BN(x_2) \\
&\approx 50 \times .5112608 - 49.9903855 \times .4918996 \\
&\approx 0.972785
\end{aligned}$$

(Note: I used pnorm() in R to compute N(), but there are obviously a lot of other ways to compute it.)

c. How much is the error made by using the approximate formula instead of Black-Scholes?

The error is $.9753292 - .972785 \approx .00265406$ or about 0.27¢ on an option worth about a dollar, less than 0.3% error.

3. Implied volatility

A stock has price \$50 and a call option with strike \$55 and a month to maturity has a price of \$0.70. What is the implied volatility of the option? Assume a riskfree rate of 1%/year.

The implied volatility is variance $= 0.1593$/year or annual standard deviation $= 0.3991$. There are a lot of ways of getting this answer: trial and error, graphing, binary search, Newton's method...