Employee Reload Options:
Pricing, Hedging, and Optimal Exercise

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February 18, 2000

Abstract

Reload options, call options whose exercise entitles the holder to new options, are compound options that are commonly issued by firms to employees. Although reload options typically involve exercise at many dates, the optimal exercise policy is simple (always exercise when in the money) and surprisingly robust to the assumptions about the underlying stock price and dividend process. As a result, we obtain general reload option valuation formulas that can be evaluated numerically. Furthermore, under the Black-Scholes assumptions with or without continuous dividends, there are even simpler formulas for prices and hedge ratios. With time vesting, valuation and optimal exercise are computed in a trinomial model, and we provide useful upper and lower bounds for the continuous-time case.

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1 Introduction

The valuation of options in compensation schemes is important for several reasons. Valuations are needed for preparing accounting statements and tax returns, and more generally for understanding what value has been promised to the employees and what remains in the firm. Furthermore, understanding the hedge ratios and the overall shape of the valuation function clarifies the manager’s risk exposure and incentives. This paper studies the optimal exercise and valuation of a particular type of employee option, the reload option, sometimes referred to as a restoration or replacement option.

The reload option has the feature that if the option is exercised prior to maturity and the exercise price is paid with previously-owned shares, the holder is entitled to one new share for each option exercised plus new options which reload or replace the original options. There is some variation on how many options are granted on exercise; here we focus on the case where the option holder receives one new reload option with strike price equal to the current price and the same expiration date as the original options for every share tendered. This provision leads to a particularly simple exercise policy (exercise whenever the option is in the money) even for dividend-paying stocks.

Our interest in reload options derives from Hemmer, Matsumaga, and Shevlin (1996), who documented the use of the various forms of the reload option in practice, demonstrated the optimal exercise policy, and valued the reload option using a binomial model for the stock price and a constant interest rate. Arnason and Jagannathan (1994) employ a binomial model to value a reload option that can be reloaded only once. Saly, Jagannathan, and Huddart (1999) value reload features under a variety of restrictions on exercise in a binomial framework. Our contribution is to provide values for the reload option for more general stochastic processes governing the interest rate, dividends and stock price, under the assumption that there is no arbitrage in complete financial markets. This is important since 1) our result does not rely on choosing a binomial approximation under which to evaluate the option, and 2) our approach yields simple valuation and hedging formulas which can be computed easily in terms of the maximum of the log of the stock price.

Our results shed light on some of the controversy about reload options. Some sensational claims about how bad reload options are have appeared in
For example, there is a suggestion that being able to exercise again and again and get new options represents some kind of money pump, or that this means that the company is no longer in control of the number of shares issued. However, even with an infinite horizon (which can only increase value compared to a finite horizon), the value of the recall option lies between the value of an American call and the stock price. Furthermore, given that the exercise price is paid in shares, the net number of new shares issued under the whole series of exercises is bounded by the initial number of reload options just as for corresponding call options. Another suggestion in the press is that the reload options might create bad incentives for risk-taking or for reducing dividends. In general, it is difficult to discuss incentives without information on other pieces of a manager’s compensation package and knowledge of what new pieces will be added and in what contingencies. However, we can reject the idea that the reload option amounts to giving the manager a short position in the equity or a put option. This argument seems to be based on the idea that the employee can “lock in” gains in the stock price by exercising the options and selling shares received upon exercise. However, upon converting the shares to cash, the employee not only locks in the gains but he also gives up some future upside. Moreover, the employee still owns options which grow in value as the stock price goes up. Indeed, we will see that the replicating portfolio is a leveraged equity position. Thus, it appears that these incentives are not so different than the corresponding incentives for employee stock options.

2 Background and No-Arbitrage Bounds

Reload options were first offered by Norwest in 1988 and were included in 17% of new stock option plans in 1997, up from 14% in 1996. Reload options are essentially American call options with an additional bonus for the holder. When exercising a reload option with a strike price of $K$ when the stock price is $S$, the holder receives one share of stock in exchange for $K$. In addition, when the strike price is paid using shares valued at current market price ($K/S$ shares per option), the holder also receives for each share tendered a new reload option of the same maturity but with a strike equal

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1See, for example, Reingold and Spiro (1998) and Gay (1999).
to the stock price at the time of tender. For example, if a manager owns 100 reload options with a strike of $100 and the stock price at time of exercise is $125, 80 shares of stock with total market value of $125 \times 80 = 10,000 are required to pay the strike price of $10,000 = $100 per option \times 100 options. Assuming frictionless buying and selling (an assumption we will maintain throughout) or at least pre-existence of shares needed to tender in the manager’s portfolio, the exercise will net 20 (= 100 - 80) shares of stock with market value of $2,500 (= $125 per share \times 20 shares), and in addition the manager will receive 80 new reload options (one for each share tendered), each having a strike price of $125 and the same maturity as the original reload options. As for other types of options issued to executives, there is some variation in reload option contracts used in practice. For example, a small proportion (about 10% according to Hemmer, Matsunaga, and Shevlin (1996)) of the options allow only a single reload, so the new options are simple call options. We analyze the more common case in which many reloads are possible. Another variation in practice is that each new option may require a vesting period before it can be exercised. We focus primarily on the simpler case in which the option can be exercised anytime after issue, but we analyze the case with vesting in Section 7. The analysis there includes numerical analysis in a trinomial model and useful bounds on the value in continuous time. Interestingly, the value under the optimal exercise policy is not much different from the value of exercising whenever the option is in the money at multiples of the vesting period.

Before proceeding to the analytic valuation of a reload option, it is useful to establish no-arbitrage upper and lower bounds on the option price. Besides developing our intuition, these bounds will help us to assess claims we have seen in the press that suggest that there is no limit to the value of a reload option that can be reloaded again and again, especially if (as is sometimes the case) the new options issued have a life extending beyond the life of the one previously exercised.

The useful lower bound on a reload option’s value is the value of an American call option. The reload option can be worth no less because the holder can obtain the American call’s payoff by following the American call’s optimal exercise strategy without ever exercising the reloaded options.

The upper bound on a reload option is the underlying stock price, no matter how many reloads are possible and no matter how long the maturity of the option, even if it is infinitely lived. This observation debunks effectively the
popular claim that not having a limit on the number of reloads or the overall maturity means that the company is losing control of how many options or shares can be generated. To demonstrate this upper bound requires a bit of analysis. Arguing along the lines of the example above, the first exercise (say at price $S_1$) yields the manager, for each reload option, $(1 - K/S_1)$ shares and $K/S_1$ new reload options with strike $S_1$. At the second exercise (say at price $S_2$), the manager nets an additional $(K/S_1)(1 - K/S_2)$ shares, for a total of $(1 - K/S_1) + (K/S_1)(1 - S_1/S_2) = (1 - K/S_2)$ shares from both exercises, and $(K/S_1)(S_1/S_2) = K/S_2$ new options with strike $S_2$. After the $i$th exercise, the manager will have in total $(1 - K/S_i)$ shares and $K/S_i$ new options with strike $S_i$. Therefore, no matter how far the stock price rises, the manager will always have fewer than one share per initial reload option, and the value is further reduced because the manager will not receive the early dividends on all of the shares. Therefore, the manager would be better off holding one share of stock and getting the dividends for all time, and the stock price is an upper bound for the value of a reload option.

Before proceeding to the formal analysis, it is worthwhile noting some obvious comparative statics. First, the value of a reload option, like the value of a call option, is decreasing in the strike price. It is increasing in the stock price for cases in which changing the stock price is a simple rescaling of the process. Given the value of the underlying investment, a higher dividend rate decreases the value of a reload, since what you get from each exercise is less. Finally, we would normally expect the value of a reload option to increase with volatility and we show that, under Black and Scholes (1973) assumptions, this is the case.  

3 Underlying Stock Returns and Valuation

Our model has two primitive assets, a locally riskless asset, “the bond,” with price process $B(t) > 0$, and a risky asset, “the stock,” with price process $S(t) > 0$. Time $t$ takes values $0 \leq t \leq T$ and all random variables and random processes are defined on a common filtered probability space.  

3This cannot be completely general for the same reasons as in Jagannathan (1984).  

4If the space is $(\Omega, \mathcal{F}, P, \{\mathcal{F}(t)\}_{t \in [0,T]})$, we denote by $E_t[\cdot]$ expectation conditional on $\mathcal{F}(t)$. All random processes are measurable with respect to this filtration. We will also consider expectations under the “risk-neutral” probability measure $P^*$ with $E_t^*$ defined
assume that that $S(t)$ is a special semimartingale that is right-continuous and left-limiting. The risky asset may pay dividends, and the nondecreasing right-continuous process $D(t) > 0$ denotes the cumulative dividend per share. We actually require very little structure on the bond price process $B(t)$; positivity and measurability is enough for most of our results and finite variation is needed for another. Of course, we would normally expect much more structure on $B(t)$; if interest rates exist and are positive then $B(t)$ would be decreasing and differentiable. For some particular valuation results (but not the proof of the optimal strategy), we will assume that $S(t)$ can only jump downwards (as it would on an ex-dividend date) but not upwards. These particular valuation results will be used to obtain a simple formula for the Black-Scholes case with or without continuous dividends.

To value a cash flow, it is equivalent to use a replicating strategy or risk-neutral valuation. While we will use risk-neutral valuation in our proofs, we look first at how a replicating strategy would work, since that clarifies our notation. Suppose we want to replicate a payoff stream whose cumulative cash flow is given by the nondecreasing right-continuous process $C(t)$. (Taking as primitive the cumulative cash flow $C(t)$ admits lumpy withdrawals as well as continuous ones. For example, choosing $C(t) = 0$ for $t < T$ and $C(T) > 0$ would correspond to a single withdrawal at the end.) To account for possible exercise at time 0, and more generally to allow for values of a random process before and after any exercise at $t$, we will use the values $0-$ or $t-$ respectively to indicate what is true just before the exercise (if any). Our usage is also consistent with using this notation for the left limit whenever the left limit is defined. For example, $C(t) - C(t-)$ denotes the amount of cash withdrawal at time $t$, whether $t > 0$ or $t = 0$. A replicating strategy is defined by two predictable processes, the number of bonds held $\alpha(t)$ and the number of shares held $\theta(t)$. The wealth process

\begin{equation}
W(t) = \alpha(t)B(t) + \theta(t)S(t)
\end{equation}

is constrained to be nonnegative and evolves according to

\begin{equation}
dW(t) = \alpha(t)dB(t) + \theta(t)dS(t) + \theta(t)dD(t) - dC(t).
\end{equation}

Stating matters this way does not rule out suicidal strategies (such as a doubling strategy run in reverse), but such strategies are not relevant once analogously to $E_t$. See Karatzas and Shreve (1991) for definitions of these terms.
we define the value of a cumulative cash flow $C(t)$ as the smallest value of $W(0−)$ in a consistent replicating strategy.

To rule out arbitrage, we could make assumptions about the underlying stock and bond processes, but instead we will simply assume the existence of a risk-neutral probability measure $P^*$, equivalent to $P$ (meaning that $P$ and $P^*$ agree on what events have positive probability), that can be used to price all assets in the economy. Under $P^*$, investing in the stock is a fair gamble in present values, and we have that for $s ≥ t$

$$\frac{S(t)}{B(t)} = E^*_t[\frac{S(s)}{B(s)} + \int_t^s \frac{1}{B(u)}dD(u)]$$

We will assume complete markets, which implies $P^*$ is unique and, moreover, it is well known that in this circumstance we can write the time $0−$ price of any consumption withdrawal stream as

$$E^*[\int_{t=0}^T \frac{1}{B(t)}dC(t)].$$

This expression is equal to $W(0−)$ in any efficient candidate replicating strategy. It is less than $W(0−)$ for a wasteful strategy that throws away money. Money could be thrown away by never withdrawing it ($W(T) > 0$) or by following a suicidal policy. The valuation in (4) is the relevant one, since we are not interested in wasteful strategies.

### 4 Reload Options with Discrete Exercise

The reload option, with strike price $K$ and expiration date $T$, is an option which, if exercised on or before the expiration date and the exercise price is paid with previously owned shares, entitles the holder to one share for each option exercised plus one new reload option per share tendered. The new reload option has a strike price equal to the current stock price and it has the same expiration date as the original option. Our basic assumption is that the option holder has unrestricted access to the financial markets; in this case the holder would always have enough shares to be able to pay the exercise price. Moreover, under this assumption, the reload option holder would be indifferent between receiving payment in cash or accumulating shares since the effects can be reversed through financial transactions. For the purpose of
computing the optimal exercise policy, it turns out to be easier to consider the latter case. In this case, we see that the payoff to exercising a single reload option with strike price $K$ at time $t \leq T$ is $1 - K/S(t)$ shares plus $K/S(t)$ new reload options with strike price $S(t)$ and expiration date $T$. Of course, the reload option holder must decide when subsequently to exercise these new options.

There is a slight technical issue concerning the definition of payoffs given the possibility of continuous exercise of reload options. To finesse this issue, we consider in this section exercise at a discrete grid of dates. The following section will consider the continuous case, for which there is a singular control that can be handled very simply by looking at well-defined limits of the discrete case. (This is analogous to the singular control of regulated Brownian motion, as in Harrison (1985).)

For the rest of this section, we assume that exercise is available only on the set of nonstochastic times $\{t_1, t_2, ..., t_n\}$, where $0 = t_1 < t_2 < ... < t_n = T$. An exercise policy is defined to be an increasing family of stopping times, $\tau_i$ taking values on the grid with $t_1 \leq \tau_1 < ... < \tau_i < ...$. Our assumption here is that the reload option holder accumulates shares and collects cash dividends from these holdings of shares. A different assumption about the disposition of the shares (for example an assumption that they are sold immediately) would not affect value, since the net value of any trade in the market is zero; we will find that a different assumption is useful for a different purpose later. In this case the number of shares received after the first exercise is $(1 - K/S(\tau_1))$ and the option holder receives $K/S(\tau_1)$ new reload options with strike price $S(\tau_1)$. The number of shares held after the exercise of the new reload options is $(1 - K/S(\tau_1)) + (K/S(\tau_1) - K/S(\tau_2)) = (1 - K/S(\tau_2))$. In general, after the $i$th exercise, the reload option holder will have accumulated $(1 - K/S(\tau_i))$ shares as well as $K/S(\tau_i)$ new reload options with strike price $S(\tau_i)$, where we set $S(\tau_0) = K$. (This is the same as the result derived in Section 3 only now in formal notation.) This simple form of the number of shares after the $i$th exercise makes it possible to write the value of this strategy as

$$E^*[\frac{S(T)}{B(T)}(1 - \frac{K}{X(T)}) + \int_0^T (1 - \frac{K}{X(t)}) \frac{1}{B(t)} dD(t)]$$
where $X(\cdot)$ is the strike or exercise price process defined by

$$X(t) = \begin{cases} K & 0 \leq t < \tau_1 \\ S(\tau_1) & \tau_1 \leq t < \tau_2 \\ S(\tau_2) & \tau_2 \leq t < \tau_3 \\ \vdots & \end{cases}$$

since the strike price is initially $K$ and later is the price of the most recent exercise.

Under our assumptions, the reload option holder’s problem is to choose an exercise policy to maximize (5). The value is increasing in the number of shares held at each time. Fortunately, the strategy of exercising whenever the reload option is in the money maximizes the number of shares at all times, and we have the following lemma which is the main result of this section.

**Lemma 1** It is an optimal policy to exercise the reload option whenever it is in the money, and hold it whenever it is out of the money. This results in the payoff

$$E^*[\frac{S(T)}{B(T)}(1 - \frac{K}{M^n(T)}) + \int_0^T \frac{1}{B(t)}(1 - \frac{K}{M^n(t)})dD(t)]$$

where

$$M^n(t) \equiv \max\{K, \max\{S(t_i)|t_i \leq t\}\}$$

is the nondecreasing process that describes the strike price as a function of time under this optimal strategy on the grid with $n$ points. This is the only optimal strategy (up to indifference about exercising at dates when the option is at the money) if the stock price can always fall between grid dates (which we think of as the ordinary case).

**Proof** Without loss of generality, assume that there is no exercise when the options are at the money (this is irrelevant for payoffs). First we show that the payoff is as claimed if we exercise at exactly those grid dates when the reload option is in the money. That follows from (5) once we show that $M^n(t) \equiv X(t)$ for the claimed optimal policy. When $t < \tau_1$, no exercise has taken place and the maximum in the definition must be $K$ (or there would have been exercise at the first date greater than $K$, contradicting $t < \tau_1$).
When $\tau_1 < t$, there has been at least one exercise. In this case, there must have been an exercise at the first date achieving the largest price so far (which is necessarily larger than $K$ or there would have been no exercise so far). And there can not have been any subsequent exercise, since the option has not been in the money since then. This shows that $M^n(t)$ is indeed the exercise price at $t$.

Now, we need to show that this is an optimal strategy. This follows trivially since the number of shares $(1 - K/M^n(t)) \geq (1 - K/X(t))$ for all $t$ and for any candidate exercise policy $X(t)$. If the stock price can always decrease between grid dates, then not following essentially this strategy reduces the value since there is positive probability of missing the maximal stock price on grid dates if we do not exercise and then the term corresponding to shares at $T$ will be smaller than under the optimum.

The Lemma admits the possibility that there are optimal strategies in which we do not exercise whenever the option is in the money, but only for the esoteric case in which it is known in advance the stock price will rise for certain.\footnote{This does not necessarily imply arbitrage if for example the stock return in the period will be either half or twice the riskfree rate.} This esoteric case is not consistent with what we know about actual stock prices, and we think of it as a mathematical curiosity. Therefore, we should think of the strategy of exercising when the option is in the money as the optimal strategy.

To study the optimal exercise strategy, we have found it useful to view proceeds of exercise as the net receipt of shares that will be held until the maturity of the option. On the other hand, for valuation and hedging, it is more useful to treat each exercise as a cash event. In other words, upon receiving shares, the reload option holder sells them at the market price. This perspective gives us the alternative valuation formula

$$\sum_{i: \tau_i \leq T} E^*[\frac{1}{B(\tau_i)} \frac{K}{X(\tau_i-)} (S(\tau_i) - X(\tau_i-))].$$

Of course, (9) and (5) have the same value for a given exercise policy. This is the subject of the next result.

**Lemma 2** Given any exercise policy, we have that the expressions (9) and (5) are the same.
Proof (sketch) Simple algebra shows that the difference between (9) and (5) is the sum over $i$ of the values of cash flows corresponding to purchase of $(K/X(\tau^*_i)) - (K/X(\tau^*_i -))$ shares at time $\tau^*_i$ and sale at time $T$, where $\tau^*_i \equiv \min(\tau_i, T)$. This is the unconditional expectation of the number of shares times the difference of the two sides of (3) for $t = \tau^*_i$ and $s = T$ but without the $E_t^*[\cdot]$. This expression, with terms for purchase, sale, and intermediate dividends, has mean zero conditional on information at $\tau^*_i$ and therefore unconditional zero expectation, which is what is to be proven.  \[ \text{Corollary 1} \]

For the optimal exercise policy in Lemma 1, we have that the optimal value (7) can be written equivalently as

\[ E^* \left[ \sum_{j=1}^{n} \frac{1}{B(t_j)} \frac{K}{M^n(t_j -)}(M^n(t_j) - M^n(t_j -)) \right] \]

Proof On dates $t_j$ when there is no exercise (i.e. $t_j \neq \tau_i$ for any $i$), $M^n(t_j) - M^n(t_j -) = 0$ and consequently the $j$th term in (10) is 0. The other dates are exercise dates, and the term in (10) equals the corresponding term in (9).

Using the formula (10) in simulations on a fine grid is probably a good way to evaluate reload options for general processes. In view of the dependence on the maximum, using the idea from Beaglehole, Dybvig, and Zhou (1997) of drawing intermediate observations from the known distribution of the maximum of a Brownian bridge should accelerate convergence significantly.

5 Valuation of Reload Options with Continuous Exercise

When the manager can exercise the reload option continuously in time, there is a technical issue of how to define payoffs. If we restrict the manager to exercising only finitely many times, we do not achieve full value, while if the

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A more formal proof of this result might use Karatzas and Shreve (1991) problem 1.2.17 and Doob’s optional sampling theorem in passing from the expectation conditional on a fixed time in (3) to the expectation conditional on the stopping time $\tau^*_i$.}

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manager can exercise infinitely many times it may not be obvious how to define the payoff. We finesse these technical issues by looking at exercise on a continuous set of times as a suitable limit of exercise on a discrete grid as the grid gets finer and finer. Given the simple form of the optimal exercise policy, this yields formulas in the continuous-time case that are just as simple as the formulas for discrete exercise. We derive these formulas in this section, and we specialize them to the Black-Scholes world in the following section.

Consider first the valuation formula (7) based on the corresponding discrete optimal strike price process (8). As the grid becomes finer and finer, the strike price process converges from below to its natural continuous-time analog

\[ M(t) \equiv \max\{K, \max\{S(s); 0 \leq s \leq t\}\} \]  

and consequently the value converges from below to its natural continuous-time version

\[ E^*\left[ S(T) \frac{1}{B(T)}(1 - \frac{K}{M(T)}) + \int_0^T \frac{1}{B(t)}(1 - \frac{K}{M(t)})dD(t) \right], \]

which is the same as (7) except with the continuous process \( M \) substituted for \( M^n \).

Consider instead the alternative formula (10). The sum in this expression can be interpreted the approximating term in the definition of a Riemann-Stieltjes integral, and in the limit we have

\[ E^*\left[ \int_0^T \frac{1}{B(t)} \frac{K}{M(t)} dM(t) \right], \]

or, setting out separately the possible jump in \( M \) at \( t = 0 \) where \( M(0) - M(0-) = (S(0) - K)^+ \), we have the equivalent expression

\[ (S(0) - K)^+ + E^*\left[ \int_0^T \frac{1}{B(t)} \frac{K}{M(t-)} dM(t) \right]. \]

At this point, we add the assumption that any jumps in the process \( S \) are downward jumps, i.e., \( S(t) - S(t-) < 0 \). This assumption implies that \( M \) is continuous: \( M \) can only jump up where \( S \) does and \( S \) cannot, while \( M \) is a cumulative maximum and therefore cannot jump down. It is nice that the assumption we need is also exactly the assumption that accomodates
predictable dividend dates (which are times when the stock price can jump down), provided reinvesting dividends results in a continuous wealth process. This assumption rules out important discrete events (for example, a merger announcement that causes the stock price to jump up 40%). From the continuity of $M$, $dM(t)/M(t-) = d\log(M)$, and defining $m(t) \equiv \log(M(t)/M(0))$ we have the alternative valuation expression

\[(15) \quad (S(0) - K)^+ + K E^*[\int_0^T \frac{1}{B(t)} dm(t)].\]

Integration by parts and interchanging the order of integration gives

\[(16) \quad (S(0) - K)^+ + K (E^*[\int_0^T \frac{1}{B(T)} m(T)] - E^*[\int_0^T m(t)\frac{1}{B(t)}]),\]

which is the formula that will allow us to derive a simple expression for the Black-Scholes case with dividends.

6 Black-Scholes Case with Dividends

In this section, we consider the Black-Scholes (1973) case with possible continuous proportional dividends. We assume a constant positive interest rate $r$, so bond prices follow

\[(17) \quad B(t) = e^{rt}.\]

With the Black-Scholes assumption of a constant volatility per unit time and continuous proportional dividends, the stock price and cumulative dividend processes follow

\[(18) \quad S(t) = S(0) \exp((\mu(t) - \frac{\sigma^2}{2} - \delta)dt + \sigma dZ(t))\]

and

\[(19) \quad D(t) = \int_0^t \delta S(u)du,\]

where $r, \sigma > 0$ and $\delta > 0$ are constants, the mean return process $\mu(t)$ is “arbitrary” (in quotes because it cannot be so wild that it generates arbitrage, e.g., by forcing the terminal stock price to a known value), and $Z(t)$ is a
standard Wiener process. Under the risk-neutral probabilities $P^*$, the form of the process is the same but the mean return on the stock is $r$.

The following Lemma gives formulas for the value and hedge ratio of the reload option. Given that there are very good uniform formulas (in terms of polynomials and exponentials) for the cumulative normal distribution function, the valuation and hedging formulas can be computed using two-dimensional numerical integration.

**Lemma 3** Suppose stock and bond returns are given by (17)–(19) (the Black-Scholes case with dividends) and the current stock price is $S(0)$. Consider a reload option with current strike price $K$ and remaining time to maturity $\tau$. Its value is

$$\frac{(S(0) - K)^+ + K(e^{-r\tau}E^*[m(\tau)] + r\int_0^{\tau} e^{-rs}E^*[m(s)]ds)}{S(0)},$$

where the cumulative distribution function of $m(t)$ is given by

$$P^*\{m(t) \leq y\} = \begin{cases} 0 & \text{for } y < 0 \\ \Phi\left(\frac{y - b - \alpha t}{\sigma \sqrt{t}}\right) - \exp\left(\frac{2\alpha(y - b)}{\sigma^2}\right)\Phi\left(-\frac{y + b - \alpha t}{\sigma \sqrt{t}}\right) & \text{for } y \geq 0 \end{cases}$$

for $y \geq 0$, where $b \equiv - (\log(K/S(0)))^+$, $\alpha \equiv r - \delta - \frac{\sigma^2}{2}$, and $\Phi(\cdot)$ is the unit normal cumulative distribution function. The reload option’s replicating portfolio holds

$$\frac{K}{S(0)} \left( e^{-r\tau}P^*(m(\tau) > 0) + r\int_0^{\tau} e^{-rs}P^*(m(s) > 0)ds \right)$$

shares. Note that this hedge ratio and the valuation formula (20) are both per option currently held, and does not adjust for the falling number of options held when there is exercise.

Before turning to the proof of Lemma 3, we direct the reader to Figures 1 and 2 which show values of reload options for various parameters, while Figures 3 and 4 compare the values of the reload option to those of a European call option. These figures confirm that the reload option value is increasing in $\sigma$ and decreasing in $\delta$. From Figure 3, we see that the reload option value for a non-dividend-paying stock is quite close to that of the European call option for low volatility but, as the volatility increases, there is a widening
Figure 1: Reload option values for various volatilities and dividend rates. This shows the value of a par reload option with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) for three different annual dividend payout rates (0, 0.2, and 0.4), assuming an annual interest rate of 5%. As for an ordinary call option, the reload’s value is increasing in volatility and decreasing in the dividend payout rate.
Figure 2: Reload option values for various interest and dividend rates
This shows the value of a par reload option with 10 years to maturity and a strike of $1.00 as a function of the interest rate (annual number) for three different annual dividend payout rates (0, 0.2, and 0.4), assuming an annual standard deviation of .2. As for an ordinary call option, the reload’s value is increasing in the interest rate and decreasing in the dividend payout rate.
Figure 3: Comparison of reload option values with a Black-Scholes European call option: no dividends
This shows the value of a par reload option (upper curve) and European call (lower curve) with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) when there are no dividends, assuming an annual interest rate of 5%. The two values move further apart as volatilities increase over the range shown, but both asymptote to $1.00 (the stock price) asymptotically.
Figure 4: Comparison of reload option values with a Black-Scholes European call option: 4% dividends
This shows the value of a par reload option (upper curve) and European call (lower curve) with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) when there are 4% annual dividends, assuming an annual interest rate of 5%. The two values move further apart more quickly than without dividends as volatilities increase, and in fact the reload asymptotes to a higher value. However, an American call would asymptote to the same value (the stock price).
spread between the reload option value and the European call value. For volatilities much larger than are shown, the two must converge again, since both converge to the stock price as volatility increases. In Figure 4, we see that for a dividend paying stock, the reload option value is uniformly higher than the European call option, as would be the value of an American call option.

The hedge ratio is always strictly positive and less than or equal to one. The hedge ratio is equal to one precisely when the option is at the money. Having a hedge ratio of ±1 at the exercise boundary is familiar for American put and call options, and is an implication of the smooth-pasting conditions. The reason for the hedge ratio of 1 in this model is also due to smooth-pasting, but is slightly more subtle to understand because both the shares we get from exercise and the new reload options contribute to the hedge ratio. If we think of delaying exercise a short while, we will have the increase/decrease in the stock price on the net number of shares we get from exercising, and we will also have the same increase/decrease on the number of reload options (since the reload options are issued at-the-money with a hedge ratio of 1). Since the number of new reload options plus the net number of new shares is equal to the number of old reload options, we can see that a hedge ratio of one is consistent with the usual smooth-pasting condition.

Proof of Lemma 3  Assume without loss of generality that \( \mu = r \), so that \( P^* = P \) and no change of measure is needed. First, note that (20) is obtained by substituting (17) into (16). (Recall that (16) assumed \( S(t) \) has no upward jumps, which is true here because \( S(t) \) defined by (18) is continuous.) Thus, we see from (20) and (18) that the value depends on the distribution of an expected maximum of a Wiener process with drift. Specifically, define \( \alpha \equiv r - \delta - \frac{\sigma^2}{2} \) and

\[
(23) \quad n(t) = \max_{0 \leq s \leq t} \log(S(s)/S(0)) = \max_{0 \leq s \leq t} (\alpha t + \sigma Z(t)).
\]

Then, \( m(t) = \max(n(t) + (\log(K/S(0)))^+, 0) \), and the distribution of \( n(t) \) is well-known: see for example Harrison (1985, Corollary 7 of Chapter 1, Section 8). Specifically, \( P\{n(t) \leq y\} = 0 \) for \( y < 0 \) and

\[
(24) \quad P\{n(t) \leq y\} = \Phi\left(\frac{y - \alpha t}{\sigma \sqrt{t}}\right) - \exp\left(\frac{2\alpha y}{\sigma^2}\right) \Phi\left(\frac{-y - \alpha t}{\sigma \sqrt{t}}\right).
\]
for \( y \geq 0 \), where \( \Phi(\cdot) \) is the unit normal distribution function. The claimed form of \( m \)'s distribution function (in (21) and associated text) follows immediately.

It remains to derive the hedging formula (22). The hedge ratio (“delta”) of the reload option is the derivative of the value, exclusive of the first term in the (20) which is received up front and not to be hedged, with respect to the stock price.\(^7\) From (20), we can see that the hedge ratio will depend on the derivative of \( E[m(t)] \) with respect to \( S(0) \). It is convenient to compute \( E[m(t)] \) using an integral over the density of \( n(t) \) since the density of \( m(t) \) depends on \( S(0) \) while the density of \( n(t) \) does not. Letting \( \Psi(y) \) be the cumulative distribution function for \( n(t) \) (given by (24) and \( \Psi(y) = 0 \) for \( y < 0 \)), we have that

\[
\frac{\partial}{\partial S(0)} E[m(t)] = \frac{\partial}{\partial S(0)} \int_{n=(\log(K/S(0)))^+}^{\infty} (n - \log(K/S(0))) d\Psi(n) = \frac{1}{S(0)} \int_{n=(\log(K/S(0)))^+}^{\infty} d\Psi(n) = \frac{1}{S(0)} P(m(t) > 0).
\]

This expression is all we need to show that (22) is the derivative of (20) exclusive of the first term with respect to \( S(0) \).

\[\blacksquare\]

7 Time Vesting

In many cases, the reload option holder is prohibited from exercising the reload option until the end of an initial vesting period. Typically, the reload options received after the initial exercise are also subject to the same vesting period. For example, reload options recently granted by Texaco have a vesting period of six months.\(^8\) We have two approaches to analyzing options

\(^7\) Some readers may be surprised to think of the hedge ratio as the simple derivative of the value with the stock price in the context of this complex seemingly path-dependent option. However, in between exercises, a reload option’s value is a function of the stock price and time, just like a call option or a European put option in the Black-Scholes world.

\(^8\) In some cases, the initial grant will vest differently than subsequent reload options. Furthermore, some firms, for example Travelers Group, have performance requirements to receive a reload option on exercise. We do not address these issues here but they can be incorporated easily into the trinomial model discussed in Section 7.
with time vesting. The indirect approach approximates the option value using a lower bound based on restricting exercise to multiples of the vesting period from maturity. The direct approach uses a trinomial model with two state variables: the moneyness of the option and the amount of time the option has been vesting. The indirect approach is simpler and may be adequate for many purposes. The direct approach can be used to compute the option value and optimal exercise boundary to arbitrary precision, but only for specific stock price processes that can be approximated by a recombining trinomial.9

A useful upper bound is the value we have obtained for continuous exercise. A useful lower bound—and also a useful approximation to the value—is the value we have obtained for discrete exercise, provided that the time interval between adjacent dates \( t_{i-1} \) and \( t_i \) is (except perhaps the first interval) equal to the vesting period. For example, if the vesting period is 6 months and we are 20 months from maturity, we assume we can exercise the option 2, 8, 14, and 20 months from now. This is a lower bound to the value given time vesting, since exercise at these dates is feasible given the vesting restriction, although it may be optimal to exercise on some other dates too. For this restrictive case, we know from Section 4 that is optimal to follow the policy of exercising whenever the reload option is in the money. Thus we can bound the value of the reload option by the unrestricted case considered previously and the value computed in the overly restricted case. We think of this lower bound (which is the value with exercise at multiples of the vesting period from the end) as being a good approximation to the actual optimum, and general in the sense that we can use simulation to compute the option values for any stock price process. To evaluate our claim that this is a pretty good approximation for most purposes, we turn to analysis that produces accurate valuation in a trinomial model.

To obtain a more accurate evaluation of a reload option with time vesting, we use a trinomial model which can be viewed as an approximation to the continuous time model. The trinomial model is similar to the binomial model of Cox, Ross, and Rubinstein (1979), but allows the stock to go up, down, or stay the same. This avoids some artificial even period/odd period

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9While it is in principle possible to add state variables to approximate any process, our experience with this model suggests that will require computers with much more memory than current computers and/or some innovation in the computation (for example from truncating parts of the tree that do not contribute much to the value).
irregularity of the exercise boundary in the binomial model.\textsuperscript{10} There are two state variables (not counting time) in our trinomial model: the ratio of the stock price to the reload’s stock price, and the number of periods of vesting. The stock price itself is not needed as a state variable, since increasing the stock price and strike price in proportion also increases the reload option value in the same proportion. To keep the number of nodes manageable, we choose up and down movements for the stock that multiply to one.

To illustrate the trinomial model, here is a sketch of how it works in the simplest case without dividends. Over the time interval $\Delta t$, the stock price is increased by a factor $u = 1 + \sigma \sqrt{3\Delta t}/2$, stays the same, or is reduced by a factor $d = 1/u$. We take the risk-neutral probability of staying the same to be $\pi^*_m = 1/3$ and we take the risk-neutral probabilities $\pi^*_u$ and $\pi^*_d$ of going up or down to be what they need to be to make the mean stock return equal to the riskless rate. As $\Delta t$ shrinks, the risk-neutral probabilities of the three states all converge to 1/3, and the variance of the stock return is approximately $\sigma^2 \Delta t$, as is desired.

Now, let $v_{i,k,m}$ be the option value per unit of strike when we are $m$ time intervals (of length $\Delta t$ each) from maturity, for the $i$th stock price node, and when we have been vested $k$ periods (with the convention that vesting of the required $K$ or more periods is labelled as $K$). At maturity ($m = 0$), the value is given by the option value. Letting $s_{i,m} = u^{i-1-m}$ be the stock price per unit of strike at node $(i, k, m)$, then we have at maturity that $v_{i,k,0} = \max(s_{i,0} - 1, 0)$. Before maturity ($m > 0$), when we are not yet vested ($k < K$), the value of the option is given by

$$v_{i,k,m} = \frac{\pi^*_uv_{i+2,k+1,m-1} + \pi^*_mv_{i+1,k+1,m-1} + \pi^*_dv_{i,k+1,m-1}}{1 + r\Delta t}.$$ 

Before maturity ($m > 0$), if we are fully vested ($k = K$), then the value is the larger of the value from exercising now or not:

$$v_{i,K,m} = \max(s_{i,m} - 1 + v_{m+1,K,m}, \frac{\pi^*_uv_{i+2,K,m-1} + \pi^*_mv_{i+1,K,m-1} + \pi^*_dv_{i,K,m-1}}{1 + r\Delta t}).$$

\textsuperscript{10}There is a conceptual difference between the binomial and trinomial models, namely that the stock and bond span all the claims in the binomial model but not in the trinomial model. However, this does not matter to us because pricing in each model converges to pricing in the continuous model as we increase the number of periods per year.
In this expression, \( s_{i,m} - 1 \) is the value we get from exercising the option, and \( v_{m+1,K,m} \) is the value of the new reload options issued at par. There is no adjustment for the number of options; the number issued (\( 1/s_{i,m} \) per option before reload) is just cancelled by the increase in strike (by a factor \( s_{i,m} \)).

The optimal exercise boundary in the trinomial model is shown in Figure 5. In viewing the picture, we should keep in mind that even with 1000 periods per year (as used in the computations for the figure), the discreteness in the price grid shows up as steps in the boundary. Qualitatively, this is what is happenening in the figure. The original option has ten years to maturity and becomes vested in 6 months (as does each reloaded option, which can be reloaded again and again). During the last six months, the option (or any reload) is not exercised, since it is equivalent to an American call option (which is never optimally exercised before maturity when the stock pays no dividends). At later dates, exercise is most attractive when there is a multiple of the vesting period left in the option's life. Early in the option's life, the prospect of subsequent exercise on dates that are not a multiple of the vesting period from the end becomes more likely, and the optimal exercise boundary becomes flatter. (It is exactly flat on early dates in the figure, but this is only because of the discreteness of the prices on the vertical axis.) We expect that the exercise boundary will be similar but smooth between multiples of the vesting period from the end in the continuous model.

The two different approaches to valuation are compared in Tables 1 and 2. In both cases, using the lower bound (computed here by Monte Carlo simulation) provides an approximation to the value that may be acceptable for some purposes, especially given that the approximation error may well be smaller than the error due to estimation error in the volatility parameter input. In the table, \( n \) denotes the number of subperiods per year; the column \( n = \infty \) comes from extrapolation and the very good theoretical approximation that the error for small \( \Delta t \) is proportional to \( \sqrt{\Delta t} \). Although we know from above that always exercising when in-the-money at multiples of the vesting period is not optimal, the quality of the approximation shows that it is nearly optimal.
This optimal exercise boundary for a vested reload option with 10 years to maturity and a six-month vesting period was computed using a trinomial model with 1000 periods per year, assuming an underlying non-dividend-paying stock with standard deviation of 30%/year and an interest rate of 5%/year. The vertical steps are due to the discrete set of possible stock prices. During the last half-year, the option is equivalent to a European call and is never exercised.
Table 1: Quality of the approximation: six-month time vesting
A no-arbitrage lower bound, computable through simulation for
general stock price processes, is a useful approximation to the
more precise but more limited trinomial model. For these num-
bers, options and their reloads vest in 6 months. The compu-
tations assume a maturity of 10 years, standard deviation of
40%/year, an annual interest rate of 5%, and no dividends. The
parameter \( n \) is the number of periods per year, and \( n = \infty \) is a
reliable extrapolation to the limit as \( n \) increases without bound.

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<th>Volatility</th>
<th>Lower Bound</th>
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<th>Trinomial ( n = 400 )</th>
<th>Trinomial ( n = \infty )</th>
<th>Upper Bound</th>
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Table 2: Quality of the approximation: one-year time vesting
This table shows the quality of the lower bound as an approxi-
mation to the option value when the vesting period is one year.
Other assumptions are as in Table 1.

<table>
<thead>
<tr>
<th>Volatility</th>
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<th>Trinomial ( n = 400 )</th>
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8 Conclusion

Employee reload options are increasingly important. It is not possible to do a definitive analysis of the impact of reload options without including the context of the rest of the compensation package, including parts that are hard to quantify including the board’s practice in granting new employee stock options in the absence of a right to reload. Nevertheless, we hope our theoretical and numerical results will help to rationalize the debate about the merits of reload options. We feel we have been successful in countering some of the wilder claims made by critics of reload options, and we have also given a scientific basis for valuation of the options for financial reporting.
References


