Problem 4. For each of the following functions, find the critical points and classify these as local max, local min, saddle point or ‘can’t tell’:

\[(1) xy^2 + x^3y - xy, \quad (2) x^2 + 6xy + y^2 - 3yz + 4z^2 + 6x + 17y - 2z. \]

Solution : (1) First order conditions are:

\[(i) y^2 + 3x^2y - y = 0, \quad (ii) 2xy + x^3 - x = 0. \]

From (i), we get: \(y = 1 - 3x^2\) or \(y = 0\). From (ii), we get: \(x = 0\) or \(x^2 = 1 - 2y\). It is easy to see that there are six critical points:

\[(x^*, y^*) = (0, 0), (0, 1), (1, 0), (-1, 0), \left( \frac{\sqrt{5}}{5}, \frac{2}{5} \right), \text{ or } \left( -\frac{\sqrt{5}}{5}, \frac{2}{5} \right). \]

The Hessian matrix of \(xy^2 + x^3y - xy\) is:

\[H = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix} \]

at \((0, 0), (0, 1), (1, 0)\) and \((-1, 0)\),

\[H = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}, \quad \text{ and } \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix} \]

respectively, it is straightforward to check that these matrix are indefinite, so \((0, 0), (0, 1), (1, 0)\) and \((-1, 0)\) are saddle points. At \(\left( \frac{\sqrt{5}}{5}, \frac{2}{5} \right)\),

\[H = \begin{pmatrix} \frac{12\sqrt{5}}{25} & \frac{2}{5} \\ \frac{2}{5} & \frac{2\sqrt{5}}{5} \end{pmatrix} \]

is positive definite, so \(\left( \frac{\sqrt{5}}{5}, \frac{2}{5} \right)\) is a local minimum. At \(\left( -\frac{\sqrt{5}}{5}, \frac{2}{5} \right)\),

\[H = \begin{pmatrix} -\frac{12\sqrt{5}}{25} & \frac{2}{5} \\ \frac{2}{5} & -2\frac{\sqrt{5}}{5} \end{pmatrix} \]

is negative definite, so \(\left( -\frac{\sqrt{5}}{5}, \frac{2}{5} \right)\) is a local maximum.
(2) First order conditions are:

\( (i) 2x + 6y + 6 = 0, \quad (ii) 6x + 2y - 3z + 17 = 0, \quad (iii) -3y + 8z - 2 = 0. \)

From (i), we get: \( (iv) x = -3y - 3. \) From (iii), we get: \( (v) z = \frac{1}{8}(3y + 2). \) Plugging \( (iv) \) and \( (v) \) into (ii), we can solve for \( y^* = -\frac{14}{137}, \) from (iv) and (v), we get: \( x^* = -\frac{369}{137}, z^* = \frac{29}{137}. \)

The Hessian matrix is:

\[
H = \begin{pmatrix}
2 & 6 & 0 \\
6 & 2 & -3 \\
0 & -3 & 8
\end{pmatrix}
\]

it is straightforward to check that \( H \) is indefinite, so \( (-\frac{369}{137}, -\frac{14}{137}, \frac{29}{137}) \) is a saddle point.

**Problem 5.** A firm’s production function is given by

\[ Q = 2L^{1/2} + 3K^{1/2} \]

where \( Q, L \) and \( K \) denote the number of units of output, labor and capital. Labor costs are $2 per unit, capital costs are $1 per unit and output sells at $8 per unit. Find the maximum profit and the values of \( L \) and \( K \) at which it is achieved.

**Solution:** The profit is:

\[ 16L^{1/2} + 24K^{1/2} - 2L - K, \]

the first order conditions are

\( (i) 8L^{-1/2} - 2 = 0, \quad (ii) 12K^{-1/2} = 1, \)

so the critical point is: \( (L^*, K^*) = (16, 144). \) Now, we check the second order condition, the Hessian matrix of the profit function is:

\[
H = \begin{pmatrix}
-4L^{-3/2} & 0 \\
0 & -6K^{-3/2}
\end{pmatrix}
\]

it is easy to see that \( H \) is negative definite, so the profit function is concave and the critical point \( (16, 144) \) is a maximum. The maximum profit is: \( 16 \times 4 + 24 \times 12 - 2 \times 16 - 144 = 176. \)