Fixed-Income Securities
Lecture 6: Symphony of In-class Exercises
Xtreme Review

Philip H. Dybvig
Washington University in Saint Louis

- Concepts
- Definitions
- Formulas
- Practice practice practice

Copyright © Philip H. Dybvig 2000
Some Central Ideas

- arbitrage: exact and approximate
- pricing using replication
- hedging using replication
- formulas and intuitions relating rates
- traditional approaches
  - matching cash flows
  - duration
  - effective duration (mean reversion, slope and twist of yield curve)
- modern option pricing tools
  - binomial model (fast, single source of noise)
  - simulation (slow, many sources of noise, not good for American)
Basic Notation

**spot rate** $r_t$: quoted at $t - 1$ for borrowing/lending from $t - 1$ to $t$

**forward rate** $f(s,t)$: quoted at $s$ for borrowing/lending from $t - 1$ to $t$

**discount factor** $D(s,t)$: price at $s$ of receiving 1 at a future date $t$

**zero-coupon rate** $z(s,t)$: yield at $s$ for a zero-coupon bond maturing at $t$

**par coupon rate** $c(s,t)$: coupon rate (= yield) quoted at $s$ for a coupon bond maturing at $t$ and trading at par

**present value** $PV$: value today of a series of future cash flows

**present value** $NPV$: value today of a series of future cash flows, less the initial price
Basic Formulas

\[ r_t = f(t - 1, t) \]

\[ D(s, t) = \frac{1}{\prod_{s=1}^{t}(1 + f(0, s))} \]

\[ f(s, t) = \frac{D(s, t - 1)}{D(s, t)} - 1 \]

\[ D(s, t) = \frac{1}{(1 + z(s, t))^{t-s}} \]

\[ z(s, t) = D(s, t)^{-(t-s)} - 1 \]

\[ c(s, T) = \frac{1 - D(s, T)}{\sum_{t=s+1}^{T} D(s, t)} \]

\[ PV = \sum_{s=1}^{t} D(0, s)c_s \]

\[ NPV = PV - P = \sum_{s=0}^{t} D(0, s)c_s \]
In-class Exercise: computations using zero-coupon bond prices

Suppose a zero-coupon bond maturing one period out (at time 1) has a price of $90 and a zero-coupon bond maturing two periods out (at time 2) has a price of $80, both per $100 of face value.

1. Compute the discount factors $D(0, 1)$ and $D(0, 2)$.
2. Compute the forward rates $f(0, 1)$ and $f(0, 2)$.
3. If we can borrow forward at 10% from one year out until two periods out, what is the arbitrage?
In-class Exercise: computations using zero-coupon rates

Suppose the yield on a one-year discount bond is 10% and the yield on a two-year zero-coupon bond is 12%.

1. Compute the discount factors $D(0, 1)$ and $D(0, 2)$.
2. Compute the forward rates $f(0, 1)$ and $f(0, 2)$.
3. What is the price of a two-year coupon bond with a face value of $500 and a coupon rate of 20%?
In-class Exercise: matching cash flows

Suppose the price is $212 for a 2-year coupon bond with face of $200 and an annual coupon (first one one year from now) of $40. Suppose also that the price is $150 for a 1-year coupon bond with face of $150 and an annual coupon (one remaining one year from now) of $15.

Remaining pension benefits in a plan having two more years to go are $95,000 one year from now and $60,000 two years from now. What replicating portfolio of the two coupon bonds covers the pension liabilities exactly? What is the price of the replicating portfolio?
Formulas connecting rates

\[ z(0, T) = \left( \frac{T}{\prod_{s=1}^{T} (1 + f(0, s))} \right)^{1/T} - 1 \approx \frac{1}{T} \sum_{s=1}^{T} f(0, s) \]

The zero-coupon rate is an average of forward rates up to that maturity.

\[ c(0, T) = \sum_{s=1}^{T} w(0, s) f(0, s) \]

where

\[ w(0, s) = \frac{D(0, s)}{\sum_{t=1}^{T} D(0, t)} \]

The par-coupon rate is a weighted average of forward rates up to that date, with more weight on the earlier maturities.
In-class Exercise: formulas connecting rates

Suppose the spot rate is 5% and the forward rate one year out is 6%. What are the one- and two-year zero-coupon rates? What are the one- and two-year par-coupon rates?
Duration formulas

traditional (Macauley) duration:

\[
duration = \frac{\sum_{t-1}^{T} c_t D(0, t)}{\sum_{s=1}^{T} c_s D(0, s)} t
\]

The discount factors \(D(0, t)\) are usually computed using either the bond's yield (i.e. \(D(0, t) \equiv 1/(1 + y)^t\)) or using the actual discount factors. Macauley duration assumes that random shocks impact forward rates equally at all maturities.

effective duration (sens = short for sensitivity):

\[
sens(\text{effective duration}) = \left(\frac{\sum_{s=1}^{T} sens(s)c_s D(0, s)}{\sum_{s=1}^{T} c_s D(0, s)}\right)
\]

For effective duration, shocks affect different forward rates differently, so the amount of interest rate exposure is no longer proportional to time-to-maturity, even for a discount bond.

the particular effective duration measure we have used:

\[
sens(duration) = \exp(-.125 * duration)
\]

\[
duration(sens) = -\log(1 - .125 * sens)/.125
\]
In-class exercise: duration and effective duration

Suppose the yield curve today is flat at 5%. Compute the duration and effective duration of a portfolio paying $100, 10 years from now, and $100, 20 years from now. Recompute the duration and effective duration assuming a flat yield curve at 10%.
Option pricing formulas

single-period:

\[ \text{Value} = R^{-1}(\pi_U V_U + \pi_D V_D) \]

expected present value computed using artificial "risk-neutral" probabilities...

Risk-neutral probabilities could be computed from the payoffs of some asset, but more commonly we make assumptions about them directly.

multi-period:

\[ \text{Value} = E^* \left[ \frac{1}{R_1} \frac{1}{R_2} \frac{1}{R_3} \ldots \frac{1}{R_T} C_T \right] \]

This formula is especially useful for simulations but can also be used in simple binomial cases without American or conversion features.
In-class exercise: binomial model

The spot interest rate is 5%. Each year it goes up by 5% (e.g. from 5% to 10%) with risk-neutral probability 1/3 or down by 2% (e.g. from 5% to 3%) with risk-neutral probability 2/3. What is the price of a 2-year interest-rate cap with a capped rate of 5% and underlying notional amount of $100 trillion?
Mean reversion and fudge factors

For mean reversion

\[ E[r_{t+1} - r_t] = k(\tau - r_t) \]

in the binomial model with equal changes \( \delta \) or \(-\delta\) in rates, set

\[
\pi_U = \frac{1}{2} + \frac{k(\tau - r_t)}{2\delta}
\]

Without mean reversion, \( k = 0 \) and \( \pi_U = 1/2 \).

fudge factors: To match actual discount factors \( D(0, t) \), modify the original model—om—as follows:

\[
R_s = R_s^{om} \frac{D(0, s-1)/D(0, s)}{D^{om}(0, s-1)/D^{om}(0, s)}
\]

or approximately

\[
r_s = r_s^{om} + f(0, s) - f^{om}(0, s)
\]
In-class exercise: capstone problem with fudge factors and mean reversion

Consider a two-year binomial model. Start with an original model in which the short riskless interest rate starts at 5\% and moves up or down by 2.5\% each period (i.e., up to 7.5\% or down to 2.5\% at the first change). The artificial probability of each of the two states at any node is determined by whatever makes mean reversion \( k \) equal to 20\% per year with a long-term mean of 5\%.

What is the price of a one-year discount bond in this original model? the two-year discount bond?

Suppose the actual one-year discount rate in the economy is 6\% and the actual two-year discount rate is 6.5\%. Compute the fudge factors and draw the tree for the adjusted interest rate process.