Basic Notation

spot rate $r_t$: quoted at $t-1$ for borrowing/lending from $t-1$ to $t$
forward rate $f(s, t)$: quoted at $s$ for borrowing/lending from $t-1$ to $t$
discount factor $D(s, t)$: price at $s$ of receiving 1 at a future date $t$
zero-coupon rate $z(s, t)$: yield at $s$ for a zero-coupon bond maturing at $t$
par coupon rate $c(s, t)$: coupon rate (= yield) quoted at $s$ for a coupon bond maturing at $t$ and trading at par
present value $PV$: value today of a series of future cash flows
present value $NPV$: value today of a series of future cash flows, less the initial price

Basic Formulas

\[ r_t = f(t-1, t) \]
\[ D(s, t) = \frac{1}{\sum_{s=1}^{t} (1 + f(0, s))} \]
\[ f(s, t) = \frac{D(s, t-1)}{D(s, t)} - 1 \]
\[ D(s, t) = \frac{1}{(1 + z(s, t))^{t-s}} \]
\[ z(s, t) = D(s, t)^{-(t-s)} - 1 \]
\[ c(s, T) = \frac{1 - D(s, T)}{\sum_{s=t}^{T} D(s, t)} \]
\[ PV = \sum_{s=1}^{T} D(0, s)c_s \]
\[ NPV = PV - P = \sum_{s=0}^{T} D(0, s)c_s \]
In-class Exercise: computations using zero-coupon bond prices

Suppose a zero-coupon bond maturing one period out (at time $t=1$) has a price of $90$ and a zero-coupon bond maturing two periods out (at time $t=2$) has a price of $80$, both per $100$ of face value.

1. Compute the discount factors $D(0, 1)$ and $D(0, 2)$.
2. Compute the forward rates $f(0, 1)$ and $f(0, 2)$.
3. If we can borrow forward at 10% from one year out until two periods out, what is the arbitrage?

In-class Exercise: computations using zero-coupon rates

Suppose the yield on a one-year discount bond is 10% and the yield on a two-year zero-coupon bond is 12%.

1. Compute the discount factors $D(0, 1)$ and $D(0, 2)$.
2. Compute the forward rates $f(0, 1)$ and $f(0, 2)$.
3. What is the price of a two-year coupon bond with a face value of $500$ and a coupon rate of 20%?

In-class Exercise: matching cash flows

Suppose the price is $212 for a 2-year coupon bond with face of $200 and an annual coupon (first one year from now) of $40. Suppose also that the price is $150 for a 1-year coupon bond with face of $150 and an annual coupon (remaining one year from now) of $15.

Remaining pension benefits in a plan having two more years to go are $95,000 one year from now and $60,000 two years from now. What replicating portfolio of the two coupon bonds covers the pension liabilities exactly? What is the price of the replicating portfolio?

Formulas connecting rates

\[ z(0, T) = \left( \prod_{s=1}^{T} (1 + f(0, s)) \right)^{1/T} - 1 \approx \frac{1}{T} \sum_{s=1}^{T} f(0, s) \]

The zero-coupon rate is an average of forward rates up to that maturity.

\[ c(0, T) = \sum_{s=1}^{T} w(0, s) f(0, s) \]

where

\[ w(0, s) = \frac{D(0, s)}{\sum_{t=1}^{T} D(0, t)} \]

The par-coupon rate is a weighted average of forward rates up to that date, with more weight on the earlier maturities.
In-class Exercise: formulas connecting rates

Suppose the spot rate is 5% and the forward rate one year out is 6%. What are the one- and two-year zero-coupon rates? What are the one- and two-year par-coupon rates?

Duration formulas

traditional (Macauley) duration:
\[
duration = \frac{\sum_{t=1}^{T} c_t D(0, t)}{\sum_{t=1}^{T} c_t D(0, s) t}
\]
The discount factors \(D(0, t)\) are usually computed using either the bond's yield (i.e. \(D(0, t) \equiv 1/(1+y)^t\)) or using the actual discount factors. Macauley duration assumes that random shocks impact forward rates equally at all maturities.

effective duration (sens = short for sensitivity):
\[
sens(\text{effective duration}) = \frac{\sum_{s=1}^{T} sens(s) c_s D(0, s)}{\sum_{s=1}^{T} c_s D(0, s)}
\]
For effective duration, shocks affect different forward rates differently, so the amount of interest rate exposure is no longer proportional to time-to-maturity, even for a discount bond.

the particular effective duration measure we have used:
\[
sens(duration) = \exp(-.125 * duration)
\]
\[
duration(sens) = -\log(1 - .125 * sens)/.125
\]

Option pricing formulas

single-period:
\[
\text{Value} = R^{-1}(\pi_U V_U + \pi_D V_D)
\]
expected present value computed using artificial "risk-neutral" probabilities...
Risk-neutral probabilities could be computed from the payoffs of some asset, but more commonly we make assumptions about them directly.

multi-period:
\[
\text{Value} = E^*[\frac{1}{R_1} \frac{1}{R_2} \frac{1}{R_3} ... \frac{1}{R_T} C_T]
\]
This formula is especially useful for simulations but can also be used in simple binomial cases without American or conversion features.
In-class exercise: binomial model

The spot interest rate is 5%. Each year it goes up by 5% (e.g., from 5% to 10%) with risk-neutral probability 1/3 or down by 2% (e.g., from 5% to 3%) with risk-neutral probability 2/3. What is the price of a two-year interest-rate cap with a capped rate of 5% and underlying notional amount of $100 trillion?

Mean reversion and fudge factors

For mean reversion

\[ \mathbb{E}[r_{t+1} - r_t] = k(r - r_t) \]

in the binomial model with equal changes \( \delta \) or \(-\delta\) in rates, set

\[ \pi_U = \frac{1}{2} + \frac{k(r - r_t)}{2 \delta} \]

Without mean reversion, \( k = 0 \) and \( \pi_U = 1/2 \).

Fudge factors: To match actual discount factors \( D(0, t) \), modify the original model—om—as follows:

\[ R_s = R_0 \frac{D(0, s - 1)/D(0, s)}{D_{om}(0, s - 1)/D_{om}(0, s)} \]

or approximately

\[ r_s = r_{om} + f(0, s) - f_{om}(0, s) \]

In-class exercise: capstone problem with fudge factors and mean reversion

Consider a two-year binomial model. Start with an original model in which the short riskless interest rate starts at 5% and moves up or down by 2.5% each period (i.e., up to 7.5% or down to 2.5% at the first change). The artificial probability of each of the two states at any node is determined by whatever makes mean reversion \( k \) equal to 20% per year with a long-term mean of 5%.

What is the price of a one-year discount bond in this original model? the two-year discount bond?

Suppose the actual one-year discount rate in the economy is 6% and the actual two-year discount rate is 6.5%. Compute the fudge factors and draw the tree for the adjusted interest rate process.