Employee Reload Options: 
Pricing, Hedging, and Optimal Exercise

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September 22, 1998

Abstract

Reload options, call options whose exercise entitles the holder to new options, are compound options that are commonly issued by firms to employees. Although reload options typically involve exercise at many dates, the optimal exercise policy is simple (always exercise when in the money) and surprisingly robust to the assumptions about the underlying stock price and dividend process. As a result, we obtain general reload option valuation formulas that can be evaluated numerically. Furthermore, under the Black-Scholes assumptions with or without continuous dividends, there are even simpler formulas for prices and hedge ratios. In the case when passage of time is required to vest each reload option, no exact valuation formula is yet available, but we provide useful upper and lower bounds.

Preliminary and Incomplete
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1 Introduction

The valuation of options in compensation schemes is important for several reasons. Valuations are needed for preparing accounting statements and tax returns, and more generally to understand what value has been promised to the employees and what remains in the firm. Furthermore, understanding the hedge ratios and the overall shape of the valuation function clarifies the manager’s risk exposure and incentives. This paper studies the optimal exercise and valuation of a particular type of employee option, the reload option.

The reload option has the feature that if the option is exercised prior to maturity and the exercise price is paid with previously-owned shares, the holder is entitled to one new share for each option exercised plus one new reload option with strike price equal to the current price and the same expiration date for every share tendered. This provision leads to a particularly simple exercise policy (exercise whenever the option is in the money) even for dividend-paying stocks.

Our interest in reload options derives from Hemmer, Matsunaga, and Shevlin (1996), who documented the use of the various forms of the reload option in practice, demonstrated the optimal exercise policy, and valued the reload option using a binomial model for the stock price and a constant interest rate. Our contribution is to provide values for the reload option for more general stochastic processes governing the interest rate and potentially dividend paying stocks, under the assumption that there is no arbitrage in complete financial markets. This is important since 1) our result does not rely on choosing a binomial approximation under which to evaluate the option, and 2) our approach yields simple valuation and hedging formulas which can be computed easily in terms of the maximum of the log of the stock price.

Our results shed light on some of the controversy about reload options. Some sensational claims about how bad reload options are have appeared in the press.\footnote{See, for example, Reingold and Spiro (1998).} For example, there is a suggestion that being able to exercise again and again and get new options represents some kind of money pump, or that this means that the company is no longer in control of the number of shares issued. However, even with an infinite horizon (which can only
increase value), the value of the recall option lies between the value of an American call and the stock price. Furthermore, given that the exercise price is paid in shares, the net number of new shares issued under the whole series of exercises is no larger than the initial number of reload options, and therefore less than for corresponding call options. Another suggestion in the press is that the reload options might create bad incentives for risk-taking or for reducing dividends. It appears that these incentives are not so different than the corresponding incentives for employee stock options.

2 Background and No-Arbitrage Bounds

Reload options were first offered by Norwest in 1988 and were included in 17% of new stock option plans in 1997, up from 14% in 1996. Reload options are essentially American call options with an additional bonus for the holder. When exercising a reload option with a strike price of $K$ when the stock price is $S$, the holder receives one share of stock in exchange for $K$. In addition, when the strike price is paid using shares valued at current market price ($K/S$ shares per option), the holder also receives for each share tendered a new reload option of the same maturity but with a strike equal to the stock price at the time of tender. For example, if a manager owns 100 reload options with a strike of $100 and the stock price at time of exercise is $125, 80 shares of stock with total market value of $125 \times 80 = $100,000 are required to pay the strike price of $100,000 = $100 per option \times 100 options. Assuming frictionless buying and selling (an assumption we will maintain throughout) or at least pre-existence of shares needed to tender in the manager’s portfolio, the exercise will net 20 (= 100 - 80) shares of stock with market value of $20,000 (= $100 per share \times 20 shares), and in addition the manager will receive 80 new reload options (one for each share tendered), each having a strike price of $125 and the same maturity as the original reload options. As is usually the case for options issued for executive compensation, there is some variation in the contracts used in practice. For example, a small proportion (about 10% according to Hemmer, Matsunaga, and Shevlin (1996)) of the options allow only a single reload, so the new options are simple call options. We analyze the more common case in which

\footnote{Reingold and Spiro (1998).}
many reloads are possible. Another variation in practice is that each new option may require a vesting period before it can be exercised. We focus primarily on the simpler case in which the option can be exercised anytime after issue, but we have some results that allow us to bound the error we are making by ignoring this feature.

Before proceeding to the analytic valuation of a reload option, it is useful to try to establish upper and lower bounds on the option price. Besides developing our intuition, these bounds will help us to assess claims we have seen in the press that suggest that there is no limit to the value of a reload option that can be reloaded again and again, especially if (as is sometimes the case) the new options issued have a life extending beyond the life of the one previously exercised.

The useful lower bound on a reload option is the value of a simple American call option. The reload option can be worth no less because the holder can elect to follow the American call option’s optimal exercise strategy and then simply choose not to exercise the new reload option.

The upper bound on a reload option is the underlying stock price, no matter how many reloads are possible and no matter how long the maturity of the option, even if it is infinitely lived. This observation debunks effectively the popular claim that not having a limit on the number of reloads or the overall maturity means that the company is losing control of how many options or shares can be generated. To demonstrate this upper bound requires a bit of analysis. Arguing along the lines of the example above, the first exercise (say at price $S_1$) yields the manager, for each reload option, $(1 - K/S_1)$ shares and $K/S_1$ new reload options with strike $S_1$. At the second exercise (say at price $S_2$), the manager nets an additional $(K/S_1)(1 - K/S_2)$ shares, for a total of $(1 - K/S_1) + (K/S_1)(1 - S_1/S_2) = (1 - K/S_2)$ shares from both exercises, and $(K/S_1)(S_1/S_2) = K/S_2$ new options with strike $S_2$. After the $i$th exercise, the manager will have in total $(1 - K/S_i)$ shares and $K/S_i$ new options with strike $S_i$. Therefore, no matter how far the stock price rises, the manager will always have fewer than one share per initial reload option, and the value is further reduced because the manager will not receive the early dividends on all of the shares. Therefore, the manager would be better off holding one share of stock and getting the dividends for all time, and the stock price is an upper bound for the value of a reload option.
Before proceeding to the formal analysis, it is worthwhile noting some obvious comparative statics. First, the value of a reload option, like the value of a call option, is decreasing in the strike price. It is increasing in the stock price for cases in which changing the stock price is a simple rescaling of the process. Given the value of the underlying investment, a higher dividend rate decreases the value of a reload, since what you get from each exercise is less. Finally, while we would normally expect the value of a reload to increase with volatility, this is less obvious, not only because of the warnings expressed by Jagannathan (1984) about the difference between increasing volatility in actual probabilities versus risk-neutral probabilities, but also because the nature of the claim is more subtle implying that value is derived from volatility of a nonlinear function of the maximum over time of the stock price.

3 Underlying Stock Returns and Valuation

Our model has two primitive assets, a locally riskless asset, “the bond,” with price process \( B(t) > 0 \), and a risky asset, “the stock,” with price process \( S(t) > 0 \). Time \( t \) takes values \( 0 \leq t \leq T \) and all random variables and random processes are defined on a common filtered probability space.\(^3\) We assume that that \( S(t) \) is a special semimartingale that is right-continuous and left-limiting. The risky asset may pay dividends, and the nondecreasing right-continuous process \( D(t) > 0 \) denotes the cumulative dividend per share. We actually require very little structure on the bond price process \( B(t) \); positivity and measurability is enough for most of our results and finite variation is needed for another. Of course, we would normally expect much more structure on \( B(t) \); if interest rates exist and are positive then \( B(t) \) would be decreasing and differentiable. For some particular valuation results (but not the proof of the optimal strategy), we will assume that \( S(t) \) can only jump downwards (as it would on an ex-dividend date) but not upwards. These particular valuation results will be used to obtain a simple formula for

\(^3\)If the space is \((\Omega, \mathcal{F}, P; \{\mathcal{F}(t)\}_{t \in [0,T]})\), we denote by \( E_t[|·|] \) expectation conditional on \( \mathcal{F}(t) \). All random processes are measurable with respect to this filtration. We will also consider expectations under the “risk-neutral” probability measure \( P^* \) with \( E_t^*[|·|] \) defined analogously to \( E_t \). See Karatzas and Shreve (1991) for definitions of these terms.
the Black-Scholes case with the possibility of continuous dividends.

To value a cash flow, it is equivalent to use a replicating strategy or risk-neutral valuation. While we will use risk-neutral valuation in our proofs, we look first at how a replicating strategy would work, since that clarifies our notation. Suppose we want to replicate a payoff stream whose cumulative cash flow is given by the nondecreasing right-continuous process $C(t)$. (Taking as primitive the cumulative cash flow $C(t)$ admits lumpy withdrawals as well as continuous ones. For example, choosing $C(t) = 0$ for $t < T$ and $C(T) > 0$ would correspond to a single withdrawal at the end.) To account for possible exercise at time 0, and more generally to allow for values of a random process before and after any exercise at $t$, we will use the values $0-$ or $t-$ respectively to indicate what is true just before the exercise (if any). Our usage is also consistent with using this notation for the left limit whenever the left limit is defined. For example, $C(t) - C(t-)$ denotes the amount of cash withdrawal at time $t$, whether $t > 0$ or $t = 0$. A replicating strategy is defined by two predictable processes, the number of bonds held $\alpha(t)$ and the number of shares held $\theta(t)$. The wealth process

$$ W(t) = \alpha(t) B(t) + \theta(t) S(t) $$

(1)

is constrained to be nonnegative and evolves according to

$$ dW(t) = \alpha(t) dB(t) + \theta(t) dS(t) + \theta(t) dD(t) - dC(t). $$

(2)

Stating matters this way does not rule out suicidal strategies (such as a doubling strategy run in reverse), but such strategies are not relevant once we define the value of a cumulative cash flow $C(t)$ as the smallest value of $W(0-)$. In a consistent replicating strategy.

To rule out arbitrage, we could make assumptions about the underlying stock and bond processes, but instead we will simply assume the existence of a risk-neutral probability measure $P^*$, equivalent to $P$ (meaning that $P$ and $P^*$ agree on what events have positive probability), that can be used to price all assets in the economy. Under $P^*$, investing in the stock is a fair gamble in present values, and we have that for $s \geq t$

$$ \frac{S(t)}{B(t)} = E_t^*[S(s) B(t) + \int_t^s \frac{1}{B(u)} dD(u)] $$

(3)
We will assume complete markets, which implies $P^*$ is unique and, moreover, it is well known that in this circumstance we can write the time $0$ price of any consumption withdrawal stream as

$$\mathbb{E}^*[\int_{t=0}^{T} \frac{1}{B(t)} dC(t)].$$

This expression is equal to $W(0)$ in any efficient candidate replicating strategy. It is less than $W(0)$ for a wasteful strategy that throws away money. Money could be thrown away by never withdrawing it ($W(T) > 0$) or by following a suicidal policy. The valuation in (4) is the relevant one, since we are not interested in wasteful strategies.

4 Reload Options with Discrete Exercise

The reload option, with strike price $K$ and expiration date $T$, is an option which, if exercised on or before the expiration date and the exercise price is paid with previously owned shares, entitles the holder to one share for each option exercised plus one new reload option with strike price equal to the current stock price and same expiration date for every share tendered. Our basic assumption is that the option holder has unrestricted access to the financial markets; in this case the holder would always have enough shares to be able to pay the exercise price. Moreover, under this assumption, the reload option holder would be indifferent between receiving payment in cash or accumulating shares since the effects can be reversed through financial transactions. For the purpose of computing the optimal exercise policy, it turns out to be easier to consider the latter case. In this case, we see that the payoff to exercising a single reload option with strike price $K$ at time $t \leq T$ is $1 - K/S(t)$ shares plus $K/S(t)$ new reload options with strike price $S(t)$ and expiration date $T$. Of course, the reload option holder must decide when subsequently to exercise these new options.

There is a slight technical issue concerning the definition of payoffs given the possibility of continuous exercise of reload options. To finesse this issue, we consider in this section exercise at a discrete grid of dates. The following section will consider the continuous case, for which there is a singular control that can be handled very simply by looking at well-defined limits of the
discrete case. (This is analogous to the singular control of regulated Brownian motion, as in Harrison (1985).)

For the rest of this section, we assume that exercise is available only on the set of nonstochastic times \( \{t_1, t_2, ..., t_n\} \), where \( 0 = t_1 < t_2 < ... < t_n = T \). An exercise policy is defined to be an increasing family of stopping times, \( \tau_i \) taking values on the grid with \( t_1 \leq \tau_1 < ... < \tau_i < ... \). Our assumption here is that the reload option holder accumulates shares and collects cash dividends from these holdings of shares. A different assumption about the disposition of the shares (for example an assumption that they are sold immediately) would not affect value, since the net value of any trade in the market is zero; we will find that a different assumption is useful for a different purpose later. In this case the number of shares received after the first exercise is \( (1 - K/S(\tau_1)) \) and the option holder receives \( K/S(\tau_1) \) new reload options with strike price \( S(\tau_1) \). The number of shares held after the exercise of the new reload options is \( (1 - K/S(\tau_1)) + (K/S(\tau_1) - K/S(\tau_2)) = (1 - K/S(\tau_2)) \). In general, after the \( i \)th exercise, the reload option holder will have accumulated \( (1 - K/S(\tau_i)) \) shares as well as \( K/S(\tau_i) \) new reload options with strike price \( S(\tau_i) \), where we set \( S(\tau_0) = K \). (This is the same as the result derived in Section 3 only now in our formal notation.) This simple form of the number of shares after the \( i \)th exercise makes it possible to write the value of this strategy as

\[
E^*[\frac{S(T)}{B(T)}(1 - \frac{K}{X(T)}) + \int_0^T (1 - \frac{K}{X(t)}) \frac{1}{B(t)} dD(t)]
\]

where \( X(\cdot) \) is the strike or exercise price process defined by

\[
X(t) = \begin{cases} 
K & 0 \leq t < \tau_1 \\
S(\tau_1) & \tau_1 \leq t < \tau_2 \\
S(\tau_2) & \tau_2 \leq t < \tau_3 \\ & \vdots
\end{cases}
\]

since the strike price is initially \( K \) and later is the price of the most recent exercise.

Under our assumptions, the reload option holder’s problem is to choose an exercise policy to maximize (5). The value is increasing in the number of shares held at each time. Fortunately, the strategy of exercising whenever the reload option is in the money maximizes the number of shares at all times, and we have the following lemma which is the main result of this section.
Lemma 1. It is an optimal policy to exercise the reload option whenever it is in the money, and hold it whenever it is out of the money. This results in the payoff

\[ E^*[\frac{S(T)}{B(T)}(1 - \frac{K}{M^n(T)}) + \int_0^T \frac{1}{B(t)}(1 - \frac{K}{M^n(t)})dD(t)] \]

where

\[ M^n(t) \equiv \max\{K, \max\{S(t_i)|t_i \leq t\}\} \]

is the nondecreasing process that describes the strike price as a function of time under this optimal strategy on the grid with \(n\) points. This is the only optimal strategy (up to indifference about exercising at dates when the option is at the money) if the stock price can always fall between grid dates (which we think of as the ordinary case).

Proof. Without loss of generality, assume that there is no exercise when the options are at the money (this is irrelevant for payoffs). First we show that the payoff is as claimed if we exercise at exactly those grid dates when the reload option is in the money. That follows from (5) once we show that \(M^n(t) \equiv X(t)\) for the claimed optimal policy. When \(t < \tau_1\), no exercise has taken place and the maximum in the definition must be \(K\) (or there would have been exercise at the first date greater than \(K\) contradicting \(t < \tau_1\)). When \(\tau_1 < t\), there has been at least one exercise. In this case, there must have been an exercise at the first date achieving the largest price so far (which is necessarily larger than \(K\) or there would have been no exercise so far). And there can not have been any subsequent exercise, since the option has not been in the money since then. This shows that \(M^n(t)\) is indeed the exercise price at \(t\).

Now, we need to show that this is an optimal strategy. This follows trivially since the number of shares \((1-K/M^n(t)) \geq (1-K/X(t))\) for all \(t\) and for any candidate exercise policy \(X(t)\). If the stock price can always decrease between grid dates, then not following essentially this strategy reduces the value since there is positive probability of missing the maximal stock price on grid dates if we do not exercise and then the term corresponding to shares at \(T\) will be smaller than under the optimum. \(\blacksquare\)
The Lemma admits the possibility that there are optimal strategies in which we do not exercise whenever the option is in the money, but only for the esoteric case in which the stock price may rise for certain. This esoteric case is not consistent with what we know about actual stock prices, and we think of it as a mathematical curiosity. Therefore, we should think of the strategy of exercising when the option is in the money as the optimal strategy.

To study the optimal exercise strategy, we have found it useful to express the exercise policy as the collection of shares. On the other hand, for valuation and hedging, it is more useful to treat each exercise as a cash event. In other words, upon receiving shares, the reload option holder sells them at the market price. This perspective gives us the alternative valuation formula

\[
\sum_{i | \tau_i \leq T} E^*[\frac{1}{B(\tau_i)} \frac{K}{X(\tau_i -)} (S(\tau_i) - X(\tau_i -))]
\]

Of course, (9) and (5) have the same value for a given exercise policy. This is the subject of the next result.

**Lemma 2** Given any exercise policy, we have that the expressions (9) and (5) are the same.

**Proof** (sketch) Simple algebra shows that the difference between (9) and (5) is the sum over \(i\) of the values of cash flows corresponding to purchase of \((K/X(\tau_i^*)) - (K/X(\tau_i^* -))\) shares at time \(\tau_i^*\) and sale at time \(T\), where \(\tau_i^* \equiv \min(\tau_i, T)\). This is the unconditional expectation of the number of shares times the difference of the two sides of (3) for \(t = \tau_i^*\) and \(s = T\) but without the \(E^*_t[\cdot]\). This expression, with terms for purchase, sale, and intermediate dividends, has mean zero conditional on information at \(\tau_i^*\) and therefore unconditional zero expectation, which is what is to be proven.\(^4\)

As a result, we have

\(^4\)A more formal proof of this result might use Karatzas and Shreve (1991) problem 1.2.17 and Doob’s optional sampling theorem in passing from the expectation conditional on a fixed time in (3) to the expectation conditional on the stopping time \(\tau_i^*\).
Corollary 1 For the optimal exercise policy in Lemma 1, we have that the optimal value (7) can be written equivalently as

\[ E^* \left[ \sum_{j=1}^{n} \frac{1}{B(t_j) M^n(t_j-)} (M^n(t_j) - M^n(t_j-)) \right] \]

Proof On dates \( t_j \) when there is no exercise (i.e. \( t_j \neq \tau_i \) for any \( i \)), \( M^n(t_j) - M^n(t_j-) = 0 \) and consequently the \( j \)th term in (10) is 0. The other dates are exercise dates, and the term in (10) equals the corresponding term in (9).

Using the formula (10) in simulations on a fine grid is probably a good way to evaluate reload options for general processes. In view of the dependence on the maximum, using the idea from Beaglehole, Dybvig, and Zhou (1997) of drawing intermediate observations from the known distribution of the maximum of a Brownian bridge should accelerate convergence significantly.

5 Valuation Of Reload Options with Continuous Exercise

When the manager can exercise the reload option at any point in time, there is a technical issue of how to define payoffs in general. If we restrict the manager to exercising only finitely many times, we do not achieve full value, while if the manager can exercise infinitely many times it may not be obvious how to define the payoff. We finesse these technical issues by looking at exercise on a continuous set of times as a suitable limit of exercise on a discrete grid as the grid gets finer and finer. Given the simple form of the optimal exercise policy, this yields formulas in the continuous-time case that are just as simple as the formulas for discrete exercise. We derive these formulas in this section, and we specialize them to the Black-Scholes world in the following section.

Consider first the valuation formula (7) based on the corresponding discrete optimal strike price process (8). As the grid becomes finer and finer, the strike price process converges from below to its natural continuous-time
\( M(t) \equiv \max \{ K, \max \{ S(s); 0 \leq s \leq t \} \} \) 

and consequently the value converges from below to its natural continuous-time version

\[
E^*[\frac{S(T)}{B(T)}(1 - \frac{K}{M(T)}) + \int_0^T \frac{1}{B(t)}(1 - \frac{K}{M(t)})dD(t)],
\]

which is the same as (7) except with the continuous process \( M \) substituted for \( M^n \).

Consider instead the alternative formula (10). The sum in this expression can be interpreted the approximating term in the definition of a Riemann-Stieltjes integral, and in the limit we have

\[
E^*[\int_0^T \frac{1}{B(t)} \frac{K}{M(t-)} dM(t)],
\]

or, setting out separately the possible jump in \( M \) at \( t = 0 \) where \( M(0) - M(0-) = (S(0) - K)^+ \), we have the equivalent expression

\[
(S(0) - K)^+ + E^*[\int_0^T \frac{1}{B(t)} \frac{K}{M(t-)} dM(t)].
\]

At this point, we add the assumption that any jumps in the process \( S \) are downward jumps, i.e., \( S(t) - S(t-) < 0 \). This assumption implies that \( M \) is continuous: \( M \) can only jump up where \( S \) does and \( S \) cannot, while \( M \) is a cumulative maximum and therefore cannot jump down. From the continuity of \( M, \frac{dM(t)}{M(t-)} = d\log(M) \), and defining \( m(t) \equiv \log(M(t)/M(0)) \) we have the alternative valuation expression

\[
(S(0) - K)^+ + K E^*[\int_0^T \frac{1}{B(t)} dm(t)].
\]

Integration by parts and interchanging the order of integration gives

\[
(S(0) - K)^+ + K \left( E^*[\frac{1}{B(T)}m(T)] - E^*[\int_0^T m(t)\frac{1}{B(t)}] \right),
\]

which is the formula that will allow us to derive a simple expression for the Black-Scholes case with dividends.
6  Black-Scholes Case with Dividends

In this section, we consider the Black-Scholes (1973) case with possible continuous proportional dividends. We assume a constant positive interest rate \( r \), so bond prices follow

\[
B(t) = e^{rt}.
\]

With the Black-Scholes assumption of a constant volatility per unit time and continuous proportional dividends, the stock price and cumulative dividend processes follow

\[
S(t) = S(0) \exp((\mu(t) - \frac{\sigma^2}{2} - \delta)dt + \sigma dZ(t))
\]

and

\[
D(t) = \int_0^t \delta S(u)du,
\]

where \( r, \sigma > 0 \) and \( \delta > 0 \) are constants, the mean return process \( \mu(t) \) is “arbitrary” (in quotes because it cannot be so wild that it generates arbitrage, e.g., by forcing the terminal stock price to a known value), and \( Z(t) \) is a standard Wiener process. Under the risk-neutral probabilities \( P^* \), the form of the process is the same but the mean return on the stock is \( r \).

The following Lemma gives formulas for the value and hedge ratio of the reload option. Given that there are very good uniform formulas (in terms of polynomials and exponentials) for the cumulative normal distribution function, the valuation and hedging formulas can be computed using two-dimensional numerical integration.

Lemma 3  Suppose stock and bond returns are given by (17)–(19) (the Black-Scholes case with dividends) and the current stock price is \( S(0) \). Consider a reload option with current strike price \( K \) and remaining time to maturity \( \tau \). Its value is

\[
(S(0) - K)^+ + K(e^{-rt} E^*[m(\tau)] + r \int_0^\tau e^{-rt} E^*[m(t)]dt),
\]
where the cumulative distribution function of \( m(t) \) is given by \( P^*\{m(t) \leq y\} = 0 \) for \( y < 0 \) and by

\[
P^*\{m(t) \leq y\} = \Phi\left(\frac{y - b - \alpha t}{\sigma \sqrt{t}}\right) - \exp\left(\frac{2\alpha(y - b)}{\sigma^2}\right) \Phi\left(-\frac{y + b - \alpha t}{\sigma \sqrt{t}}\right)
\]

for \( y \geq 0 \), where \( b \equiv -(\log(K/S(0)))^+ \), \( \alpha \equiv r - \delta - \frac{\sigma^2}{2} \), and \( \Phi(\cdot) \) is the unit normal cumulative distribution function. The reload option’s replicating portfolio holds

\[
\frac{K}{S(0)} \left( e^{-r\tau} P^*(m(\tau) > 0) + r \int_0^\tau e^{-rs} P^*(m(s) > 0)ds \right)
\]

shares. Note that this hedge ratio and the valuation formula (20) are both per option currently held, and does not adjust for the falling number of options held when there is exercise.

Before turning to the proof of Lemma 3, we direct the reader to Figures 1 and 2 which show values of reload options for various parameters, while Figures 3 and 4 compare the values of the reload option to that of a European call option. (Warning: in this preliminary draft, the numerical results are only accurate to within 1% or so. Later drafts will have better accuracy.) These figures reveal the easily verified facts that the reload option value is increasing in \( \sigma \) and decreasing in \( \delta \). From Figure 3, we see that the reload option value for a non-dividend-paying stock is quite close to that of the European call option for low volatility but, as the volatility increases, there is a widening spread between the reload option value and the European call value. For volatilities much larger than are shown, the two must converge again, since both converge to the stock price as volatility increases. In Figure 4, we see that for a dividend paying stock, the reload option value is uniformly higher than the European call option, as would be the value of an American call option.

**Proof of Lemma 3** Assume without loss of generality that \( \mu = r \), so that \( P^* = P \) and no change of measure is needed. First, note that (20) is obtained by substituting (17) into (16). (Recall that (16) assumed \( S(t) \) has no upward jumps, which is true here because \( S(t) \) defined by (18) is continuous.) Thus, we see from (20) and (18) that the value depends on
Figure 1: Reload option values for various volatilities and dividend rates. This shows the value of a par reload option with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) for three different annual dividend payout rates (0, 0.2, and 0.4), assuming an annual interest rate of 5%. As for an ordinary call option, the reload’s value is increasing in volatility and decreasing in the dividend payout rate.
Figure 2: Reload option values for various interest and dividend rates. This shows the value of a par reload option with 10 years to maturity and a strike of $1.00 as a function of the interest rate (annual number) for three different annual dividend payout rates (0, 0.2, and 0.4), assuming an annual standard deviation of .2. As for an ordinary call option, the reload’s value is increasing in the interest rate and decreasing in the dividend payout rate.
Figure 3: Comparison of reload option values with a Black-Scholes European call option: no dividends
This shows the value of a par reload option (upper curve) and European call (lower curve) with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) when there are no dividends, assuming an annual interest rate of 5%. The two values move further apart as volatilities increase over the range shown, but both asymptote to $1.00 (the stock price) asymptotically.
Figure 4: Comparison of reload option values with a Black-Scholes European call option: 4% dividends

This shows the value of a par reload option (upper curve) and European call (lower curve) with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) when there are 4% annual dividends, assuming an annual interest rate of 5%. The two values move further apart more quickly than without dividends as volatilities increase, and in fact the reload asymptotes to a higher value. However, an American call would asymptote to the same value (the stock price).
the distribution of an expected maximum of a Wiener process with drift. Specifically, define \( \alpha \equiv r - \delta - \frac{\sigma^2}{2} \) and

\[
(23) \quad n(t) \equiv \max_{0 \leq s \leq t} \log(S(s)/S(0)) = \max_{0 \leq s \leq t} \alpha t + \sigma Z(t).
\]

Then, \( m(t) = \max(n(t) + (\log(K/S(0)))^+, 0) \), and the distribution of \( n(t) \) is well-known: see for example Harrison (1985, Corollary 7 of Chapter 1, Section 8). Specifically, \( P\{n(t) \leq y\} = 0 \) for \( y < 0 \) and

\[
(24) \quad P\{n(t) \leq y\} = \Phi\left( \frac{y - \alpha t}{\sigma \sqrt{t}} \right) - \exp\left( \frac{2\alpha y}{\sigma^2} \right) \Phi\left( \frac{-y - \alpha t}{\sigma \sqrt{t}} \right)
\]

for \( y \geq 0 \), where \( \Phi(\cdot) \) is the unit normal distribution function. The claimed form of \( m \)'s distribution function (in (21) and associated text) follows immediately.

It remains to derive the hedging formula (22). The hedge ratio ("delta") of the reload option is the derivative of the value, exclusive of the first term in the (20) which is received up front and not to be hedged, with respect to the stock price. From (20), we can see that the hedge ratio will depend on the derivative of \( E[m(t)] \) with respect to \( S(0) \). It is convenient to compute \( E[m(t)] \) using an integral over the density of \( n(t) \) since the density of \( m(t) \) depends on \( S(0) \) while the density of \( n(t) \) does not. Letting \( \Psi(y) \) be the cumulative distribution function for \( n(t) \) (given by (24) and \( \Psi(y) = 0 \) for \( y < 0 \)), we have that

\[
(25) \quad \frac{\partial}{\partial S(0)} E[m(t)] = \frac{\partial}{\partial S(0)} \int_{n=(\log(K/S(0)))^+}^{\infty} (n - \log(K/S(0))) d\Psi(n) = \frac{1}{S(0)} \int_{n=(\log(K/S(0)))^+}^{\infty} d\Psi(n) = \frac{1}{S(0)} P(m(t) > 0).
\]

This expression is all we need to show that (22) is the derivative of (20) exclusive of the first term with respect to \( S(0) \). \[\Box\]

\(^{5}\)Some readers may be surprised to think of the hedge ratio as the simple derivative of the value with the stock price in the context of this complex seemingly path-dependent option. However, in between exercises, a reload option’s value is a function of the stock price and time, just like a call option or a European put option in the Black-Scholes world.
Figure 5: Reload option bounds for time vesting: no dividends
This shows the value of a par reload option with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) for immediate vesting, vesting after 6 months, vesting after one year, and vesting after two years, assuming an annual interest rate of 5% and no dividends. The bound is a lower bound based on being able to exercise only at points in time spaced by the vesting time.
Figure 6: Reload option bounds for time vesting: 4% dividends
This shows the value of a par reload option with 10 years to maturity and a strike of $1.00 as a function of the volatility (annual standard deviation) for immediate vesting, vesting after 6 months, vesting after one year, and vesting after two years, assuming an annual interest rate of 5% and no dividends. The bound is a lower bound based on being able to exercise only at points in time spaced by the vesting time.
7 Time Vesting

In many cases, the reload option holder is prohibited from exercising the reload option for a period of time after the option is granted. Of course, the reload options received after the initial exercise are also subject to time vesting. While we do not know the optimal strategy for this case, we can bound the value with time vesting above and below. A useful upper bound is the value we have obtained for continuous exercise, and a useful lower bound is the value we have obtained for discrete exercise, provided that the time interval between adjacent dates $t_{i-1}$ and $t_i$ is least as long as the vesting period. Specifically, we restrict the reload option holder to exercise only on the vesting dates. For example, if the vesting period is six months, we partition the time to maturity into six month intervals and restrict the reload option holder to only exercise at these times. For this overly restrictive case, it is optimal to follow the policy of exercising whenever the reload option is in the money as before. However, the true value will be higher than those computed for the overly restricted case. Thus we can bound the value of the reload option by the unrestricted case considered previously and the value computed in the overly restricted case. Figures 5 and 6 illustrate these bounds for various time vesting restrictions. Here, Figure 5 illustrates these bounds for a stock which does not pay dividends. Observe that for low volatility, these bounds are fairly tight (indeed they are close to the value of the European call option). As the volatility increases, however, the bounds degrade significantly. For short vesting periods on the order of six months, the bounds are tighter than for longer vesting periods. For vesting periods more than two years, these bounds are not all that much different than the value of the European call option. Figure 6 illustrates these bounds for a stock with a fairly high dividend yield. Here, the bounds are wider due to the difference in the value of dividends.

8 Conclusion

In this paper we valued and computed the hedging portfolio for the employee reload option. We derive exact valuation formulas. These expressions can be explicitly and easily calculated for some examples without having to
construct binomial trees. Naturally, the usual caveats for valuing employee options apply to our results. Probably the most bothersome is the issue that we take the stock and dividend processes to be exogenously specified. It would be difficult to relax this assumption. On the other hand, it seems that various imperfections related to the reload option holder’s access to financial markets are less bothersome than in other types of employee compensation issues, at least for the optimal exercise policy, since the employee is quite likely to be holding shares in their place of employment. However, certain holding period or exercise restrictions may change the nature of the optimal exercise decision. The simplest case is where the reload option, upon exercise entitles the holder to shares and an ordinary warrant. This is an optimal stopping problem which can be easily evaluated on a binomial tree using backwards induction, but it is easy to see that the optimal exercise policy will differ from the case we analyze here.
References


