Modeling Term Structures of Defaultable Bonds

Duffie and Singleton (1999)

Presented by Xicheng Jiang
Outline

• Introduction of defaultable claims modeling
• Consider alternative recovery methods
• Valuation of defaultable bonds
Review

• For a contingent claim $X$ at $T$, given its real-world cont’d return $\mu$:
  \[ V_0 = e^{-\mu T} E[X] \]

• Using the equivalent martingale approach:
  \[ V_0 = e^{-r T} E^Q [X] \]

• If the risk-free rate $r$ is random process (this is the case in most fixed-income modelling)
  \[ V_0 = E^Q \left[ \exp \left( - \int_0^T r_t \, dt \right) X \right] \]
Hazard Rate

• Survival function: $S(t) = \text{prob}(T > t)$, which is decreasing
• Default probability: $S(t) - S(t + \Delta t) = \text{prob}(t \leq T < t + \Delta t)$
• Conditional default probability:
  \[
  \frac{S(t) - S(t + \Delta t)}{S(t)} = \frac{\text{prob}(t \leq T < t + \Delta t)}{\text{prob}(T > t)} = \text{prob}(T < t + \Delta t | T > t)
  \]
• “Density” of conditional default probability: $\frac{S(t) - S(t + \Delta t)}{\Delta t \cdot S(t)}$
• Hazard rate: $h(t) = \lim_{\Delta t \to 0} \frac{S(t) - S(t + \Delta t)}{\Delta t \cdot S(t)}$
• As a result, the conditional default probability in a short time interval $dt$ can be written as $h(t)dt$
Intuition

• Short rate process $r_t$ and equivalent martingale measure $Q$
• Let $h_t$ denotes the hazard rate for default at time $t$
• Let $L_t$ denotes the expected fractional loss in market value if default were to occur at time $t$, conditional on $\mathcal{F}_t$
• The initial market value of the defaultable claim to $X$ is

$$V_0 = E^Q \left[ \exp \left(- \int_0^T R_t \, dt \right) X \right]$$

where the default-adjusted short-rate process $R_t = r_t + h_t L_t$
• Need to be proven under both discrete and continuous settings
Defaultable Claims in Discrete Space

• Let $\varphi_s$ denotes the dollar amount of recovery given default at time $s$. What’s the market value of an asset $V_t$, given future recovery $\varphi_{t+1}$ given default and future value $V_{t+1}$ given no default?

$$V_t = h_t e^{-r_t E_t^Q [\varphi_{t+1}]} + (1 - h_t) E_t^Q [V_{t+1}]$$

• Recursively solving forward...

$$V_t = E_t^Q \left[ \sum_{j=0}^{T-1} h_{t+j} e^{-\sum_{k=0}^{j} r_{t+k} \varphi_{t+j+1}} \prod_{l=0}^{j} (1 - h_{t+l-1}) \right] + E_t^Q \left[ e^{-\sum_{k=0}^{T-1} r_{t+k} \varphi_{t+T}} \prod_{j=1}^{T} (1 - h_{t+j-1}) \right]$$
Defaultable Claims in Discrete Space

• Suppose we adapt “RMV” (recovery of market value) assumption here, i.e., take the RN expected recovery as a fraction of RN expected survival contingent market value.

\[ E_s^Q[\varphi_{s+1}] = (1 - L_s)E_s^Q[V_{s+1}] \]

• Substitute it into the \( V_t \) expression:

\[ V_t = h_te^{-r_t}(1 - L_t)E_t^Q[V_{t+1}] + (1 - h_t)e^{-r_t}E_t^Q[V_{t+1}] \]

\[ = E_t^Q[e^{-\sum_{j=0}^{T-1} R_{t+j} X_{t+T}}] \]

• Where \( e^{-R_t} = (1 - h_t)e^{-r_t} + h_t e^{-r_t} (1 - L_t) \)

• Or \( R_t = r_t + h_t L_t \)
Defaultable Claims in Discrete Space

• Why this representation is good?

• If we assume that $h_t$ and $L_t$ are exogenous process, we can just model $R$ for the defaultable bonds, instead of $r$, using single- or multifactor model such as CIR or Vasicek, or HJM model.

• State Dependence is accommodated, i.e., $h_t$ and $L_t$ may be correlated with each other, with $r_t$, with economic cycle...

• If the exogeneity is violated, we must find other methods. (For example, market value of recovery is fixed..)
Figure 1
Distributions of recovery by seniority
Defaultable Claims in Continuous Space

- Contingent claim \((Z, \tau)\): random variable \(Z\) and stopping time \(\tau\) where \(Z\) is paid. \(Z\) is \(\mathcal{F}_\tau\) measurable.

- The ex-dividend price process \(U\) for \((Z, \tau)\) is given by:
  \[
  U_t = E_t^Q \left[ \exp \left( - \int_t^\tau r_u du \right) Z \right]
  \]

- Defaultable claim \(((X, T), (X', T'))\): \((X, T)\) is the obligation of issuer to pay \(X\) at \(T\). \((X', T')\) defines the stopping time \(T'\) when the issuer defaults and \(X'\) is recovered.

- Actual claim \((Z, \tau)\) generated by such a defaultable claim is defined by:
  \[
  \tau = \min(T, T') \quad Z = X 1_{\{T < T'\}} + X' 1_{\{T \geq T'\}}
  \]
Defaultable Claims in Continuous Space

• Note that $T'$ is random by nature since we don’t know when the issuer defaults.

• We model $T'$ by setting a variable $\Lambda_t = 1_{\{t \geq T'\}}$

• From the definition of hazard rate, we know that the instantaneous conditional default probability can be written as $h_t \, dt$. However, in this case, a defaultable claim can only default once. Once it defaults the probability will become one and will never change. To model this, we rewrite the probability as $(1 - \Lambda_t) h_t \, dt$

• After adding a demean process $M_t$, we can get

  $$d\Lambda_t = (1 - \Lambda_t) h_t \, dt + dM_t$$
Defaultable Claims in Continuous Space

• The payoff $X'$ at default is also random. It is modeled as

$$X' = (1 - L_t)U_{t-}$$

• where $U_{t-} = \lim_{s \uparrow t} U_s$ is the price of the claim "just before" default

• Key assumption is that this $L_t$ is predictable by the information up to $t$, i.e., $\mathcal{F}_t$
Defaultable Claims in Continuous Space

• We know that if we discounted the gain from an asset by the short-rate process \( r \), the gain process must be a martingale under \( Q \).
• The discounted gain process \( G \) is defined by:

\[
G_t = \exp \left( - \int_0^t r_s ds \right) V_t (1 - \Lambda_t) + \int_0^t \exp \left( - \int_0^s r_u du \right) (1 - L_s) V_{s-} d\Lambda_s
\]

• This is a martingale. What should we do to get \( V_t \)? Ito’s Lemma.
• Let \( dG_t = 0 \) and use the fact that \( X' = (1 - L_t) U_{t-} \), we can get

\[
V_t = \int_0^t R_s V_s ds + m_t
\]
Defaultable Claims in Continuous Space

• Given the terminal boundary condition $V_T = X$, we can get:

$$V_0 = E^Q \left[ \exp \left( - \int_0^T R_t dt \right) X \right]$$

where $R_t = r_t + h_t L_t$

• We can see another advantage of this model. In its final form, we can get rid of $(X', T'), \Lambda_t$ and $U_t$. That being said, we don’t need to model the characteristics of the defaultable claim. Instead, only by considering the non-defaultable contingent claim and changing the discount rate can get the final answer.
Some Special Cases

• Continuous-time Markov formulation: Assume a state variable process $Y$ that is Markovian

$$J(Y_t, t) = E^Q \left[ \exp \left( - \int_t^T \rho(Y_s)ds \right) g(Y_T) | Y_t \right]$$

• $Y_t = (Y_{1t}, Y_{2t}, ..., Y_{nt})'$ solves a SDE:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t$$

• $J$ solves the backward Kolmogorov PDE:

$$J_t(y, t) + J_y(y, t)\mu(y) + \frac{1}{2} \text{trace} \left( J_{yy}(y, t)\sigma(y)\sigma'(y) \right) - \rho(y)J(y, t) = 0$$

with boundary condition

$$J(y, T) = g(y)$$
Some Special Cases

• Price-dependent expected loss rate:

\[ J(Y_t, t) = E^Q \left[ \exp \left( - \int_t^T \rho(Y_s, J(Y_s, s)) ds \right) g(Y_T) \mid Y_t \right] \]

Corresponding PDE can be treated numerically

• Uncertainty about recovery:

\[ X' = (1 - l)U_{T'} \]

where \( l \) is a bounded, \( \mathcal{F}_{T'} \) measurable random variable

• \( L_t \) is the expectation of \( l \) given all info up to but not including time \( t \).

• \( L_{T'} = E(l \mid \mathcal{F}_{T'} \mid \_ ) \)

• With this change, the pricing formula \( R_t = r_t + h_t L_t \) still applys.
Defaultable Bonds: Recovery and valuation

• Consider the following recovery methods:

  \textbf{RT:} \ \varphi_t = (1 - L_t)P_t, \text{ where } L \text{ is an exogenously specified fractional recovery process and } P_t \text{ is the price at time } t \text{ of an otherwise equivalent, default-free bond [Jarrow and Turnbull (1995)].}

  \textbf{RFV:} \ \varphi_t = (1 - L_t); \text{ the creditor receives a (possibly random) fraction } (1 - L_t) \text{ of face ($1$) value immediately upon default [Brennan and Schwartz (1980) and Duffee (1998)].}

• Under RT, the computational burden of directly computing } V_t \text{ can be substantial. Time of default, the joint } F_t \text{-conditional distributions of } L_v, h_s, r_u \text{ for all } v, s, u \text{ between } t \text{ and } T \text{ plays a computationally challenging role in determining } V_t.
Defaultable Bonds: Recovery and valuation

- RMV: $E^Q_s [\phi_{s+1}] = (1 - L_s) E^Q_s [V_{s+1}]$

- RMV vs RFV: RMV matched to the legal structure of swap contract better. RMV model is more convenient for corporate bonds because we can just apply standard default-free term-structure modelling techniques. RFV, on the other hands, is more realistic when absolute priority applies.

- Is there a significant difference between RMV and RFV model?

- For simplicity, we take $L_t = \bar{L}$, a constant. We model $r$ and $h$ by

$$r_t = \rho_0 + Y_t^1 + Y_t^2 - Y_t^0$$

$$h_t = b Y_t^0 + Y_t^3$$

where $Y_t^i$ is “square root diffusions” under $Q$
Defaultable Bonds: Recovery and valuation

- Under RMV assumption:

\[
V^{RMV}_{nt} = cE^Q_t \left( \sum_{j=1}^{2n} e^{-\int_{t}^{t+s_j} R_s \, ds} \right) + E^Q_t \left( e^{-\int_{t}^{t+n} R_s \, ds} \right)
\]

where \( R_t = r_t + h_t \bar{L} \)

- Under RFV assumption:

\[
V^{RFV}_{nt} = cE^Q_t \left( \sum_{j=1}^{2n} e^{-\int_{t}^{t+s_j} (r_s + h_s) \, ds} \right) + E^Q_t \left( e^{-\int_{t}^{t+n} (r_s + h_s) \, ds} \right) + \int_{t}^{t+n} (1 - \bar{L}) \gamma(Y_t, t, s) \, ds,
\]

where \( \gamma(Y_t, t, s) = E^Q_t \left( h_s e^{-\int_{t}^{s} (r_u + h_u) \, du} \right) \)
Defaultable Bonds: Recovery and valuation

• Calibrate the RMV and RFV model:
  • Bonds with fixed ten-year par-coupon spreads. (known $c$)
  • Fixed $L_t = \bar{L}$
  • $r_t$ and $h_t$ are modelled by several square-root diffusion processes
  • Minimizing the error between model estimated bond prices and real bond prices through changing the parameters of $r_t$ and $h_t$.
  • Compute the mean implied intensity $\bar{h}$
Figure 2
For fixed ten-year par-coupon spreads, $S$, this figure shows the dependence of the mean hazard rate $\bar{h}$ on the assumed fractional recovery $1 - \bar{L}$. The solid lines correspond to the model $RFV$, and the dashed lines correspond to the model $RMV$. 
Figure 3
Term structures of par-coupon yield spreads for RMV (dashed lines) and RFV (solid lines), with 50% recovery upon default, a long-run mean hazard rate of $\theta_0 = 200$ bp, a mean reversion rate of $\kappa = 0.25$, and an initial hazard-rate volatility of 100%.
Noncallable Corporate Bonds

• Note that the hazard rate process $h_t$ and the fractional loss $L_t$ enter the discount rate in the product from $h_t L_t$

• Knowledge of defaultable bond prices before default alone is not sufficient to separately identify $h_t$ and $L_t$

• If one has prices of undefaulted junior and senior bonds of the same issuer, along with the prices of the Treasury bonds, we can extract $h_t L^J_t$ and $h_t L^S_t$, thus can infer $L^J_t / L^S_t$.

• We can just model jointly the dynamic properties of $r_t$ and the “short spread” $s_t \equiv h_t L_t$
Noncallable Corporate Bonds

• Case 1: Square root diffusion model of $Y$

$$r_t = \delta_0 + \delta_1 Y_{1t} + \delta_2 Y_{2t} + \delta_3 Y_{3t}$$

$$s_t = \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t} + \gamma_3 Y_{3t}$$

• Dai and Singleton (1998) proposes the “most flexible” affine term structure model

$$dY_t = \mathcal{K} (\Theta - Y_t) dt + \sqrt{S_t} dB_t$$

where $\mathcal{K}$ is a 3*3 matrix with positive diagonal and nonpositive off-diagonal elements; $\Theta$ in $\mathbb{R}_+^3$; $S_t$ is 3*3 diagonal matrix with diagonal elements $Y_{1t}, Y_{2t}$ and $Y_{3t}$
Noncallable Corporate Bonds

• Duffie (1999) considered the special case in which \( \delta_0 = -1 \) and \( \delta_3 = 0 \), so \( r_t \) could take on negative values and depend only on the first two state variables.

• He also assumed that \( \mathcal{K} \) is diagonal (\( Y_{1t}, Y_{2t} \) and \( Y_{3t} \) are independent)

• However, the only means of introducing negative correlation among \( r_t \) and \( s_t \) is to allow for negative \( \gamma \), which means the hazard rate may take on negative values.

• Within this correlated square-root model of \( (r_t, s_t) \), one cannot simultaneously have a nonnegative hazard rate process and negatively correlated \( r_t \) and \( h_t \) without having one or more \( \gamma \) or \( \delta \) negative.
Noncallable Corporate Bonds

• Case 2: More flexible correlation structure for \((r_t, s_t)\)

\[
\begin{align*}
  r_t &= \delta_0 + \delta_1 Y_{1t} + Y_{2t} + Y_3t \\
  s_t &= \gamma_0 + \gamma_1 Y_{1t} + \gamma_2 Y_{2t}
\end{align*}
\]

• We assume that

\[
\begin{align*}
  dY_t &= \mathcal{K}(\Theta - Y_t)dt + \Sigma \sqrt{S_t} dB_t
\end{align*}
\]

where \(\mathcal{K}\) is a 3*3 matrix with positive diagonal and nonpositive off-diagonal elements; \(\Theta\) in \(\mathbb{R}^3_+\); and

\[
\begin{align*}
  S_{11}(t) &= Y_1(t), \\
  S_{22}(t) &= [\beta_2]_2 Y_2(t), \\
  S_{33}(t) &= \alpha_3 + [\beta_3]_1 Y_1(t) + [\beta_3]_2 Y_2(t)
\end{align*}
\]
Noncallable Corporate Bonds

• All of $\delta_0, \delta_1, \gamma_0, \gamma_1, \gamma_2$ are strictly positive

• Dai and Singleton show that in this case the most flexible and admissible affine term structure has:

\[
K = \begin{bmatrix}
\kappa_{11} & \kappa_{12} & 0 \\
\kappa_{21} & \kappa_{22} & 0 \\
0 & 0 & \kappa_{33}
\end{bmatrix}
\quad \Sigma = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\sigma_{31} & \sigma_{32} & 1
\end{bmatrix}.
\]

• The short-spread rate $s_t$ is strictly positive. At the same time, the signs of $\sigma_{31}$ and $\sigma_{32}$ are unconstrained, so the third state variable may have increments that are negatively correlated with the first two.
Valuation of defaultable callable bonds

• During the time window of callability, we have the recursive pricing formula

\[ V_t = \min[\bar{V}_t, e^{-R_t} E_t^Q (V_{t+1} + d_{t+1})] \]

• Outside the callability window,

\[ V_t = e^{-R_t} E_t^Q (V_{t+1} + d_{t+1}) \]
Valuation of defaultable callable bonds

• In a more continuous time context, let $\mathcal{T}(t, T)$ denote the set of feasible call policies. The market price at $t$ is:

$$V_t = \min_{\tau \in \mathcal{T}(t, T)} E_t^Q \left[ \sum_{t < T(i) \leq \tau} \gamma_{t, T(i)} c_i + \gamma_{t, \tau} \right]$$

where

$$\gamma_{t, s} = \exp \left( - \int_t^s R_u \, du \right)$$

• This equation can be solved by a discrete algorithm similar to the equations in the previous slide.
More (in the paper)

• Defaultable HJM model
• Pricing Credit Derivatives
• Etc.
Take home message

• The initial market value of the defaultable claim to $X$ is

$$V_0 = E^Q \left[ \exp \left( - \int_0^T R_t \, dt \right) X \right]$$

where the default-adjusted short-rate process $R_t = r_t + h_t L_t$

• All financial products with defaultable nature can be modeled in this way.

• If we assume that $h_t$ and $L_t$ are exogenous process, we can just model the process $R$ for the defaultable bonds instead of $r$. 
My remarks (pros)

- Very detailed.
- A variety of models regards to defaultable claims/bonds under different assumptions are given.
- These models can be directly applied in the market using market data (calibration).
- Some brief comparisons of models are given.
My remarks (cons)

• Too much theoretical stuffs, should give more calibration (I mean to be theoretical is good, but it is always better to give some empirical results)

• More like a encyclopedia instead of a paper. (always introduce a model and say “please refer to some other papers”. I think a paper should focus one or a few models and dig deeper).

• Without a clear conclusion. (which model is good or bad under which conditions?)
Thank you

• 😊