FIN642 Practice Problem Set 5
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1. (Vasicek term structure model) We work entirely in the risk-neutral probabilities. The process $Z_t = (Z_{1t}, ...Z_{Nt})$ is an N-dimensional standard Wiener process. The instantaneous spot rate follows the process

$$r_t = \overline{r} + \sum_{n=1}^{N} \left(f_{n0} e^{-\kappa_n t} + \sigma_n \int_{s=0}^{t} e^{-\kappa_n (t-s)} dZ_{ns} \right)$$

where $\overline{r}$ is the long-term mean interest rate, σ_n is the local standard deviation of the nth factor, κ_n is the rate of mean reversion in the nth factor, and f_{n0} is the initial condition for the nth factor.

A. Note that if $h(t)$ is a function of time alone, then $\int_{s=0}^{T} h(s) dZ_s$ is distributed normally with mean 0 and variance $\int_{s=0}^{T} (h(s))^2 ds$. (Intuitively, this is normally distributed because it is a linear combination of joint normally distributed random variables, and the mean and variance follow from the linear isometry.) Given the information at time 0, what is the distribution of r_t?

B. Note that if $H(t)$ is a function of time alone, then $\int_{t=0}^{T} \int_{s=0}^{t} H(s) dZ_{ns} dt = \int_{s=0}^{T} \left(\int_{t=s}^{T} H(t) dt \right) dZ_{ns}$. Compute the discount bond price (price at 0 of receiving \$1 at T)

$$D_{0,T} \equiv E[\exp(-\int_{t=0}^{T} r_t dt)].$$

C. For $n = 1, ..., N$, define $f_{nt} \equiv f_{n0} e^{-\kappa_n t} + \sigma_n \int_{s=0}^{t} e^{-\kappa_n (t-s)} dZ_{ns}$. Write df_{nt} in terms of f_{nt}, dZ_{nt}, dt, and the parameters. Argue why $f_t = (f_{1t}, ..., f_{Nt})$ is a state variable for the interest rate process.

D. For arbitrary fixed s and T, $s < T$, write the discount factor $D_{s,T}$ (the price at s of receiving \$1 for sure at T)

$$D_{s,T} \equiv E_s[\exp(-\int_{t=s}^{T} r dt)].$$

in terms of the state variable vector f_s.