OPTIONS and FUTURES
Lecture 3: Put Options and Distribution-Free Results

Philip H. Dybvig
Washington University in Saint Louis

- put options
- binomial valuation
- what are distribution-free results?
- option pricing bounds
- early exercise?
- put-call parity

European put options

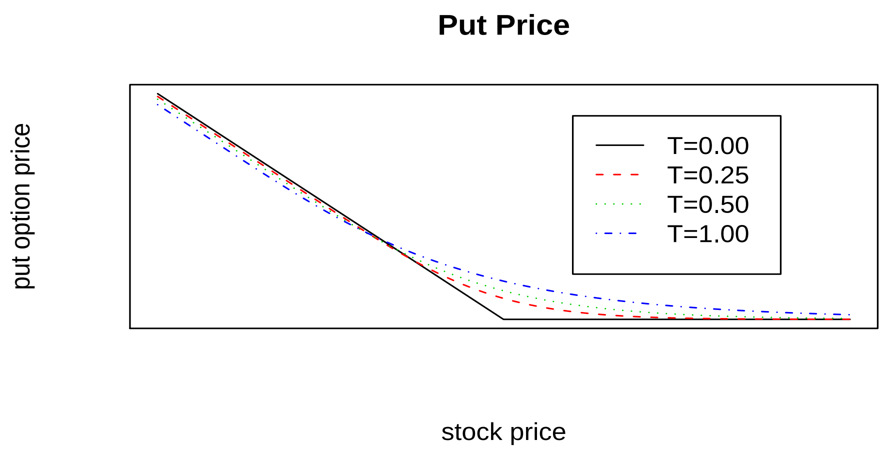
The purchaser obtains the right to sell one share of stock to the issuer at the maturity date for the pre-specified strike or exercise price.

If the strike price exceeds the stock price at maturity, the option is *in the money*. In this case, it is optimal for the owner to exercise the option. The option is worth the difference between the stock price and the strike price, which is the amount the owner pockets by exercising the option.

If the stock price exceeds the strike price at maturity, the option is *out of the money*. In this case, it is optimal for the owner to let the option expire without exercising it, because exercise would cost the owner the difference between the stock price and the strike price. The price of the option is zero.

Summary: At maturity (T), the value is $\max(0, X - S_T)$, or the maximum of zero and the exercise price less the stock price.

An *American put option* is just like the European option, but the option of exercising is available anytime (once only) at or before the maturity date.


Put Price
T=0.00
T=0.25
T=0.50
T=1.00
put option price
stock price

More on put options

The put option pays off when the underlying stock goes down but does not obligate the owner when the underlying stock goes up. For this privilege, the purchaser pays a price ($premium$) up front. The market price of the option depends on the exercise price, the stock price, the time to maturity, the volatility of the underlying stock, the riskless interest rate and the anticipated size of dividends before maturity.

As for the call, the most important influence on a put's value day-to-day is the stock's price, and the next most important influence is the volatility of the underlying stock price.

Sensitivities of options' market price

When this ↑	call value	put value
Stock price	↑	↓
Strike price	↓	↑
Time to maturity	↑	↕
Volatility	↑	↑
Interest rate	↑	↓
Dividend	↓	↑

In-class exercise: European put option

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 5/4$. What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of a European put option with a \$95 strike price maturing in two periods?

stock

put option

reminder: $\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}$.

Binomial model strategy: European and American options

Binomial valuation follows the steps

- Compute the tree for the underlying.
- Evaluate the option at the end based on the contract.
- Step back through the tree to compute the value one node at a time.

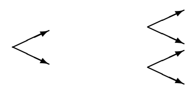
For a European option, the valuation step takes the risk-neutral expected present value looking one-period forward. For an American option, there is also a possibility of exercising the option at the node. As a consequence, the value is the larger of the value of live option (which is the risk-neutral expected present value looking one-period forward).

In-class exercise: American put option

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 5/4$. What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of an American put option with a \$95 strike price maturing in two periods?

stock

put option



reminder: $\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}$.

And now for something completely different: Distribution-free results

So far, we have been using the binomial model to perform valuation. In the binomial model, we make strong explicit assumptions about underlying stock-price movements so we can compute the option prices exactly (that is, exactly if our assumptions are true). For the coming slides we are going to consider *distribution-free* results that depend only on the absence of arbitrage (or related concepts) and place little or no assumptions on the distribution of the underlying.

Arbitrage and related concepts

Definition An *arbitrage opportunity* is a strategy that never costs us anything now or in the future, in any contingency, but has a positive probability of having a positive cash flow at some date (or dates).

Definition The *law of one price* says that assets promising the same cash flows will have the same price.

Definition *Dominance*: One set of actions dominates another if it leaves us better off at some time in some contingency and never any worse off.

An upper bound on the call option price

Throughout, assume there are no splits or assessments. There may be dividends except as noted.

Suppose there is no arbitrage. Then the value of a European call option is not larger than the stock price.

Proof: Suppose not. Then consider the strategy of buying a share of stock, selling a call option, tendering the stock if the option is exercised, and selling the stock if the option expires unexercised. This strategy pays the difference in the prices up front and never costs anything. Furthermore, any dividends before exercise or maturity, the exercise price paid on exercise, and sale of stock when the option expires unexercised are all potential additional payoffs. Therefore, this strategy is an arbitrage opportunity, which is a contradiction.

An upper bound on the call option price: worksheet

Suppose that the call price now (C) is larger than the stock price now (S). Let S_M denote the stock price at maturity of the option.

time	now	maturity
write a call		
buy stock		
net cash flow		

A lower bound on the European call option price

Suppose there is no arbitrage and it is known that the stock will pay no dividends before the option's maturity. Then the value of a European call option is no smaller than the stock price less the present value of obtaining the exercise price at the option's maturity.

Proof: Suppose not. Then consider the strategy of buying the option and *always** exercising it, selling short the stock, buying a riskless bond that promises the exercise price at the option's maturity, and pocketing the remainder (which is positive by supposition). At maturity of the option, this is a wash, because the bond will pay for the exercise of the option and the share of stock received from exercise will undo the short position. Because there are no dividends, there are no other cash outflows associated with this strategy. Therefore the strategy is an arbitrage opportunity, which is a contradiction.

*not an optimal strategy, but good enough to run the arbitrage

A lower bound on the European call option price: worksheet

Suppose $C < S - Xe^{-r_f T}$.

time	now	maturity
buy a call		
sell stock		
lend money		
net cash flow		

In-class exercise: put option valuation bound

Supposing there is no arbitrage, show that the value of a European put option is no smaller than the present value of the strike price less the stock price. Assume a continuously compounded interest rate $r_f > 0$ so \$1 now grows to $\$\exp(r_f T) > \$1$ at option maturity.

time	now	maturity
buy/sell put		
buy/sell stock		
borrow/ lend		
net cash flow		

hint: set up the arb that would be present if the European put price P is smaller than $Xe^{-r_f T} - S$ where X is the strike and S the stock price.

Relation between European and American call options

Suppose investors use undominated strategies and the law of one price holds. Also assume it is known that a stock will pay no dividends before the option's maturity. Then exercising an American call option on the stock before maturity is a dominated strategy, and therefore European and American call options have the same value.

Proof: Exercising an American call option now before maturity is dominated by pursuing the following alternative strategy. Today, obtain the same value by shorting the stock, pocketing the excess of the proceeds over the exercise price, and investing the exercise price in interest–bearing bonds. At the maturity of the option, exercise no matter what. The share of stock obtained from exercise offsets the short position. The bond principal covers the exercise price, and the interest is pocketed. Because this interest is the only difference in payoffs between the two strategies, the alternative strategy dominates exercising early. This means the two types of options have the same cash flows (since the exercise decision at maturity is the same), and therefore the law of one price implies the prices are the same.

Put-call parity theorem

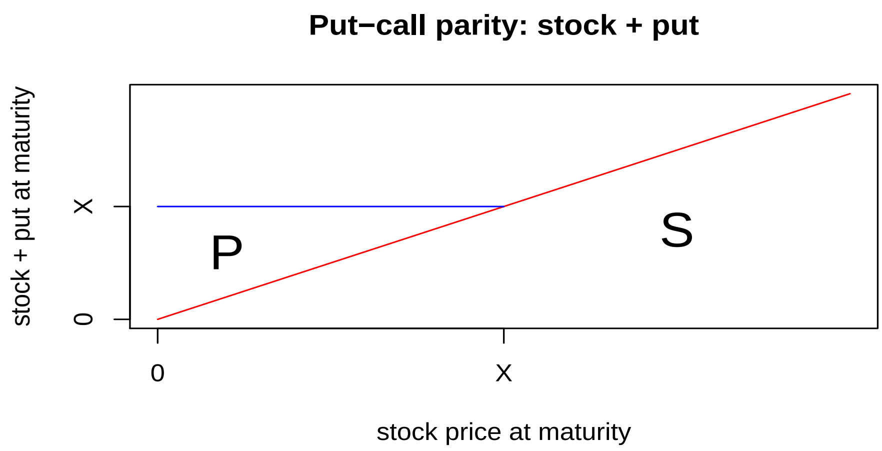
Consider two European put and call options on a a stock that pays no dividends. We assume that the options are matched in the sense that they have the same maturity date and the same exercise price. Consider also a riskless discount bond maturing at the same date as the options and having a face value equal to the common exercise price of the options.

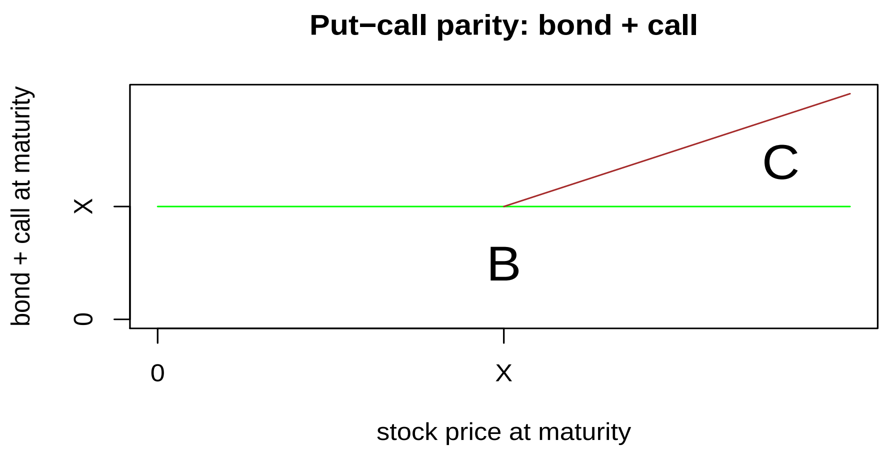
The value of holding the stock and the put is the same as the value of holding the bond and the call, i.e., for all t,

$$S_t + P_t = B_t + C_t.$$

Proof: Because we are working with European options, the value today depends only on the value at maturity. If $S_T \geq X$, $P_T = 0$, $C_T = S_T - X$, and $B_T = X$. Therefore, both $S_T + P_T$ and $B_T + C_T$ equal S_T.

For $S_T < X$, $P_T = X - S_T$, $C_T = 0$, and $B_T = X$. Therefore, both $S_T + P_T$ and $B_T + C_T$ equal X.


Put−call parity: stock + put
stock + put at maturity
X
0
P
S
0
X
stock price at maturity


Put−call parity: bond + call
bond + call at maturity
X
0
C
B
0
X
stock price at maturity

In-class exercise: put-call parity

Hi-Tech Biceps is a start-up company selling electronic exercise gear. HB is not currently paying dividends, nor are dividends expected for the next two years. HB stock now sells for $\$55$ a share, and at–the–money HB European call options maturing one year from now are selling for $\$18$ apiece. The current riskless rate is 10%. (This is an ordinary interest rate, not a continuously compounded rate.) What is a fair price for at-the-money HB European put options maturing one year from now?