OPTIONS and FUTURES
Lecture 2: Binomial Option Pricing and Call Options

Philip H. Dybvig
Washington University in Saint Louis

- binomial model
- replicating portfolio—single period
- artificial (risk-neutral) probabilities
- replicating portfolio—multiple periods
- call options

The binomial option pricing model

The option pricing model of Black and Scholes revolutionized a literature previously characterized by clever but unreliable rules of thumb. The Black-Scholes model uses continuous-time stochastic process methods that interfere with understanding the simple intuition underlying these models. We will start instead with the binomial option pricing model of Cox, Ross, and Rubinstein, which captures all of the economics of the continuous time model but is simple to understand and program. For option pricing problems not appropriately handled by Black-Scholes, some variant of the binomial model is the usual choice of practitioners since it is relatively easy to program, fast, and flexible.

Cox, John C., Stephen A. Ross, and Mark Rubinstein (1979) "Option Pricing: A Simplified Approach" *Journal of Financial Economics* **7**, 229–263

Black, Fischer, and Myron Scholes (1973) "The Pricing of Options and Corporate Liabilities" *Journal of Political Economy* **81**, 637–654

Binomial process (3 periods)

Riskless bond:

$$1 \longrightarrow r \longrightarrow r^2 \longrightarrow r^3$$

Stock ($u > r > d$):

$$\begin{tikzcd}
 & & & u^3S \\
 & & u^2S \arrow[ur] \arrow[dr] & \\
 & uS \arrow[ur] \arrow[dr] & & u^2dS \\
S \arrow[ur] \arrow[dr] & & udS \arrow[ur] \arrow[dr] & \\
 & dS \arrow[ur] \arrow[dr] & & ud^2S \\
 & & d^2S \arrow[ur] \arrow[dr] & \\
 & & & d^3S
\end{tikzcd}$$

Derivative security (option or whatever):

$$\begin{tikzcd}
 & & & V_1 \\
 & & ? \arrow[ur] \arrow[dr] & \\
 & ? \arrow[ur] \arrow[dr] & & V_2 \\
? \arrow[ur] \arrow[dr] & & ? \arrow[ur] \arrow[dr] & \\
 & ? \arrow[ur] \arrow[dr] & & V_3 \\
 & & ? \arrow[ur] \arrow[dr] & \\
 & & & V_4
\end{tikzcd}$$

What is the price of the derivative security?

A simple option pricing problem in one period

Riskless bond (interest rate is 0): $100 \longrightarrow 100$

Stock:

$$50 \quad \begin{array}{l} \nearrow 100 \\ \searrow 25 \end{array}$$

Derivative security (at-the-money call option):

$$? \quad \begin{array}{l} \nearrow 50 \\ \searrow 0 \end{array}$$

To duplicate the call option with α_S shares of stock and α_B bonds:

$$\begin{aligned} 50 &= 100\alpha_S + 100\alpha_B \\ 0 &= 25\alpha_S + 100\alpha_B \end{aligned}$$

Therefore $\alpha_S = 2/3$, $\alpha_B = -1/6$, and the duplicating portfolio is worth $50\alpha_S + 100\alpha_B = 100/6 = 16\ 2/3$. By absence of arbitrage, this must also be the price of the call option.

In-class exercise: one-period contingent claim valuation

Compute the duplicating portfolio and the price of the general derivative security below. Assume $u > r > d > 0$.

Riskless bond:

$$1 \longrightarrow r$$

Stock:

$$\begin{tikzcd}[row sep=small] & u \\ 1 \arrow[ur] \arrow[dr] & \\ & d \end{tikzcd}$$

Derivative security:

$$\begin{tikzcd}[row sep=small] & V_u \\ ??? \arrow[ur] \arrow[dr] & \\ & V_d \end{tikzcd}$$

Multi-period valuation and artificial probabilities

In general, exactly the same valuation procedure is used many times, taking as given the value at maturity and solving back one period of time until the beginning. This valuation can be viewed in terms of state prices p_u and p_d or risk-neutral probabilities π_u and π_d, which give the same answer (which is the only answer consistent with no arbitrage):

$$Value = p_u V_u + p_d V_d = r^{-1}(\pi_u V_u + \pi_d V_d)$$

where

$$p_u = r^{-1}\frac{r-d}{u-d} \quad p_d = r^{-1}\frac{u-r}{u-d}$$

and

$$\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}.$$

Interpretation: all investments are fair gambles, subject to discounting to account for impatience and a probability adjustment to account for risk pricing.

In-class exercise: digital option

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 1$. What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of a *digital option* that pays \$100 when the final stock price is greater than \$120 and pays \$0 otherwise.

stock

digital option

hint: Start by filling in the stock value tree. Then compute the option values at the end. Finally, use single-period valuation to step back through the tree one node at a time.
reminder: $\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}$.

Some orders of magnitude

- Expected excess returns
  - Common stock indices: $4–5\%$ per year or $4–5\%/250 \approx 1.5$ or 2 basis points daily
  - Individual common stocks: $50\%–150\%$ of the index expected excess return.
- Standard deviation
  - Common stock indices: $15–20$% per year or $15–20$%$/\sqrt{250} \approx 0.75–1\%$ per day
  - Individual common stocks: $35\%–40\%$ per year

Theoretical observation: for the usual case, standard deviations over short periods are almost exactly the same in actual probabilities as in risk-neutral probabilities.

Digression: review of interest rates and compounding

Simple interest: the amount paid for use of money is Prt, where P is principal, r is the interest rate, and t is the time interval (most commonly measured in years). For example, the interest on $\$200$ for 2 months at a 12% annual rate is $\$200 \times 12\% \times 2/12 = \$4$.

Compound interest: interest is recorded periodically and added to the principal; consequently there is interest on the interest. For example, if 12% interest is compounded semi–annually and we start with $\$1000$, then after three years we will accumulate $\$1000 \times (1 + 12\%/2)^{2\times 3} \approx \$1418.52$.

Continuous compounding: the value in the limit as interest is recorded more and more frequently is given by the exponential function. For example, if 12% interest is compounded continuously and we start with $\$1000$, then after three years we will have $\$1000 \times \exp(12\% \times 3) = \$1000e^{0.36} \approx \$1433.33$.

Over short horizons, all three are almost exactly the same!

Binomial parameters in practice

Most texts seem to have unreasonably complicated expressions for u, d, and r in binomial models. From the theory, we know that a good choice is

$$u = 1 + r * \Delta t + \sigma * \sqrt{\Delta t}$$

$$d = 1 + r * \Delta t - \sigma * \sqrt{\Delta t}$$

with $\pi_u = \pi_d = 1/2$ and Δt the time increment. This has the two essential features: it equates risk-neutral expected stock and bond returns, and it has the right standard deviation. In addition, it has a continuous stock price (like Black-Scholes) as a limit.

One alternative is to choose the positive solution of $ud = 1$ and $u - d = 2\sigma\sqrt{\Delta t}$ (good for option price as a function of time) or a recombining trinomial (good for including some dependence of variance on the stock price).

European call option

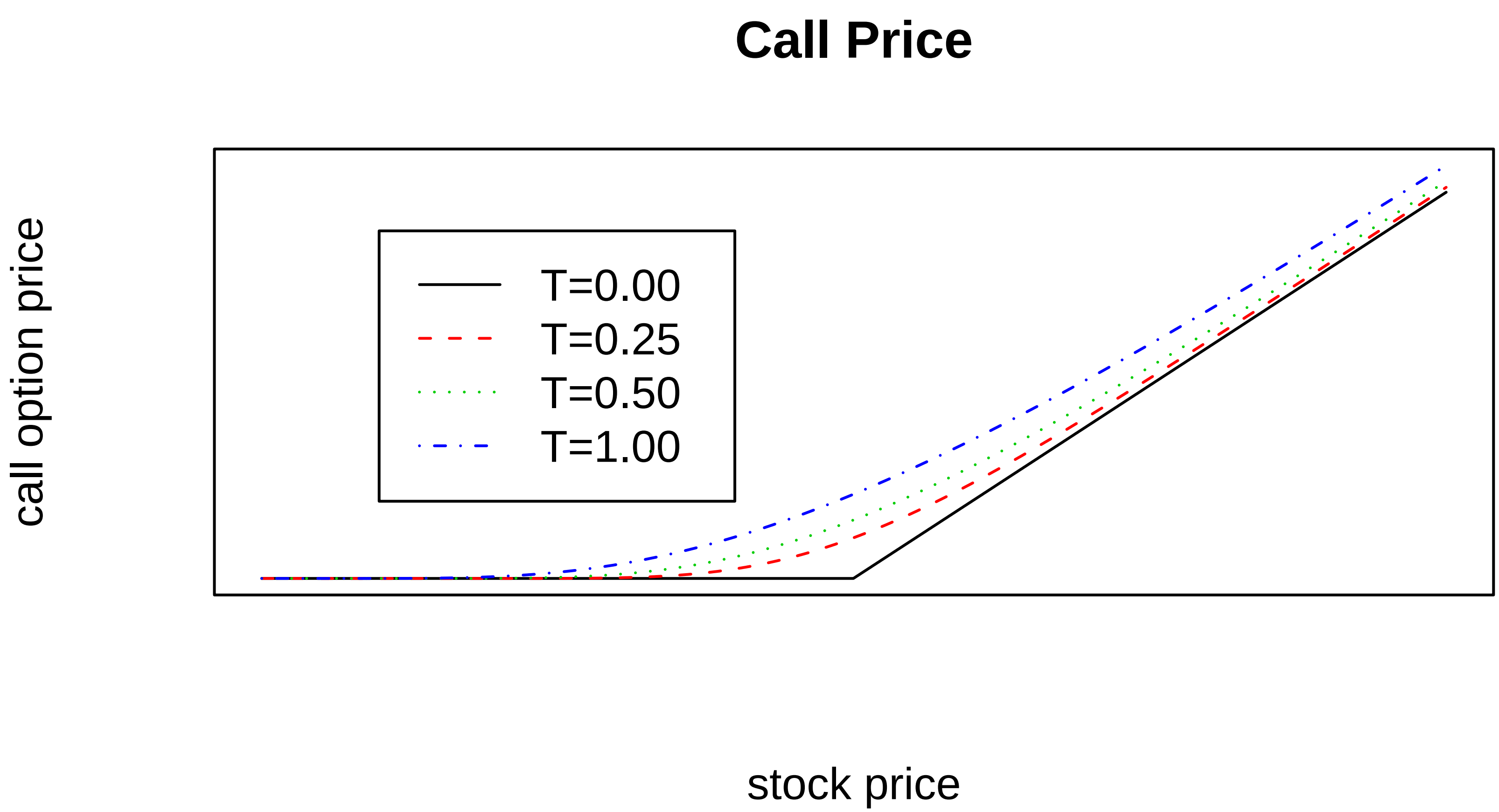
The purchaser obtains the right to buy one share of stock from the issuer at the maturity date for the pre-specified strike or exercise price.

If the stock price exceeds the strike price at maturity, the option is $\textit{in the money}$. In this case, it is optimal for the owner (the purchaser) to exercise the option. The option is worth the difference between the stock price and the strike price, which is the amount the owner pockets by exercising the option.

If the strike price exceeds the stock price at maturity, the option is $\textit{out of the money}$. In this case, it is optimal for the owner to let the option expire without exercising it, because exercise would cost the owner the difference between the stock price and the strike price. The price of the option is zero.

Summary: At maturity (T), the value is $\max(S_T - X, 0)$, or the maximum of the stock price less the exercise price and zero.

An $\textit{American call option}$ is just like the European option, but the option of exercising is available anytime (once only) at or before the maturity date.


Call Price
T=0.00
T=0.25
T=0.50
T=1.00
call option price
stock price

More on call options

The call option pays off when the underlying stock goes up but does not obligate the owner when the underlying stock goes down. For this privilege, the purchaser pays a price (also called a *premium*) up front. The market price of the option depends on the exercise price, the stock price, the time to maturity, the volatility of the underlying stock, the riskless interest rate and the anticipated size of dividends before maturity.

The most important influence on an option's value day-to-day is the stock's price. The next most important influence on the option's value is the volatility of the underlying stock price. In fact, one interesting use of option prices is to compute *implied volatilities*, that is the level of volatility ("vol") implicit in the option price. The other influences either do not change on a daily basis (the exercise price and the anticipated remaining dividends) or have movements with relatively minor effect on the option price at normal maturities (time to maturity and the riskless interest rate).

In-class exercise: call option valuation

Consider the binomial model with $u = 2$, $d = 1/2$, and $r = 1$. What are the risk-neutral probabilities? Assuming the stock price is initially \$100, what is the price of a call option with a \$90 strike price maturing in two periods?

stock

call option

reminder: $\pi_u = \frac{r-d}{u-d} \quad \pi_d = \frac{u-r}{u-d}$.

Wrap-up

Binomial pricing of options!

- single-period: use risk-neutral probabilities and discounted expected value
- multiple-period: use single period valuation again and again
- strategy:
  - generate a valuation tree for the underlying
  - value the option at the terminal date based on contract terms
  - use single-period valuation to step through the tree
- call option example