Suggested Solutions to Homework 6

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Problem 1. (i) Suppose that the *p.d.f.* of a certain random variable X has the following form:

$$f(x) = \begin{cases} cx & 0 < x < 4 \\ 0 & \text{otherwise,} \end{cases}$$

where c is a given constant. Determine the value of c and also the values of $Pr(1 \leq X \leq 2)$ and $Pr(X > 2)$.

Solution : (i) From

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^4 cxdx = 1,$$

we get $c = \frac{1}{8}$.

$$Pr(1 \leq X \leq 2) = \int_1^2 \frac{1}{8}xdx = \frac{3}{16},\ and\ Pr(X > 2) = \int_2^4 \frac{1}{8}xdx = \frac{3}{4}.$$

(ii) Suppose that a random variable X has a uniform distribution on the interval $[-2, 8]$, find (i) the *p.d.f.* of X; (ii) the value of $Pr(0 < X < 7)$; (iii)the mean and the variance of X.

Solution : *p.d.f.* of X is

$$f(x) = \frac{1}{10},\ and\ Pr(0 < x < 7) = \int_0^7 \frac{1}{10}dx = \frac{7}{10}.$$

$$E(X) = \int_{-2}^{8} \frac{x}{10} = 3,\ Var(X) = \int_{-2}^{8} (x-3)^2 \frac{1}{10}dx = \frac{25}{3}.$$

Problem 2 Suppose that the joint *p.d.f.* of two random variables X and Y is as follows:

$$f(x, y) = \begin{cases} c(x^2 + y) & 0 \leq y \leq 1 - x^2 \\ 0 & \text{otherwise.} \end{cases}$$

Determine (i) the value of the constant c; (ii) $Pr(0 \leq X \leq \frac{1}{2})$; (iii) $Pr(Y \leq X + 1)$.

Solution : (i)

$$\int_{-1}^{1}\int_{0}^{1-x^2} c(x^2+y)dydx = 1,$$

we get:

$$c\int_{-1}^{1}(x^2(1-x^2)+\frac{1}{2}(1-x^2)^2)dx = 1,$$

we get: $\frac{c}{2}(x-\frac{1}{5}x^5)|_{-1}^{1} = 1$, so $c = \frac{5}{4}$.

(ii)

$$Pr(0 \le X \le \frac{1}{2}) = \int_{0}^{\frac{1}{2}}\int_{0}^{1-x^2}\frac{5}{4}(x^2+y)dydx = \frac{5}{8}(x-\frac{1}{5}x^5)|_{0}^{\frac{1}{2}} = \frac{79}{256}.$$

(iii)

$$Pr(Y \le X+1) = \int_{-1}^{0}\int_{0}^{x+1}\frac{5}{4}(x^2+y)dydx + \int_{0}^{1}\int_{0}^{1-x^2}\frac{5}{4}(x^2+y)dydx$$

$$= \frac{5}{4}\left(\frac{1}{4}x^4+\frac{1}{3}x^3+\frac{1}{6}(x+1)^3\right)|_{-1}^{0} + \frac{5}{8}(x-\frac{1}{5}x^5)|_{0}^{1} = \frac{13}{16}.$$

<u>Problem 3</u> (i) Suppose that X and Y are independent Possion random variables such that $Var(X)+Var(Y) = 5$. Evaluate $P(X+Y<2)$.
(ii) Suppose that X_1 and X_2 are independent random variables, and X_i has an exponential distribution with parameter $\beta_i (i=1,2)$. Find $Pr(X_1 > kX_2)$, where $k>0$ is a constant.

Solution : (i) Assume

$$P(X=x) = \frac{e^{-\lambda_1}\lambda_1^x}{x!}, x = 0,1,2.., \; P(Y=y) = \frac{e^{-\lambda_2}\lambda_2^x}{y!}, y = 0,1,2..,$$

From $Var(X)+Var(Y) = 5$, we get: $\lambda_1+\lambda_2 = 5$.

$$Pr(X+Y<2) = Pr(X=0,Y=1)+Pr(X=0,Y=0)+Pr(X=1,Y=0)$$

$$= e^{-\lambda_1-\lambda_2}(\lambda_1+\lambda_2+1) = 0.0404.$$

(ii)

$$f(x_1) = \frac{1}{\beta_1}e^{-\frac{x_1}{\beta_1}}, x_1 > 0, f(x_2) = \frac{1}{\beta_2}e^{-\frac{x_2}{\beta_2}}, x_2 > 0,$$

$$P(x_1 > kx_2) = \int_0^\infty \int_{kx_2}^\infty \frac{1}{\beta_1\beta_2}e^{-\left(\frac{x_1}{\beta_1}+\frac{x_2}{\beta_2}\right)}dx_1dx_2 = \frac{\beta_1}{\beta_1+k\beta_2}.$$

Problem 4 Let X and Y have a continuous distribution with joint $p.d.f.$

$$f(x,y) = \begin{cases} x+y & 0 \le x \le 1 \text{ and } 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the covariance $Cov(X,Y)$.

Solution :

$$E(XY) = \int_0^1\int_0^1 xy(x+y)dxdy = \int_0^1 (\frac{1}{3}y + \frac{1}{2}y^2)dy = \frac{1}{3},$$

$$E(X) = \int_0^1\int_0^1 x(x+y)dxdy = \int_0^1 (\frac{1}{3} + \frac{1}{2}y)dy = \frac{7}{12},$$

and

$$E(Y) = \int_0^1\int_0^1 y(x+y)dxdy = \int_0^1 (\frac{1}{2}y + y^2)dy = \frac{7}{12}.$$

Therefore, $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$.