Fin 500J: Suggested Solutions to Homework 3

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Problem 1. Consider the problem of maximizing $f(x, y, z) = xyz + z$, subject to the constraints $x^2 + y^2 + z \leq 6, x \geq 0, y \geq 0, z \geq 0$.
(1) Write out a complete set of first order conditions for this problem.
(2) Determine whether or not the constraint $x^2+y^2+z \leq 6$ is binding at any solution.
(3) Find a solution of the first order conditions that includes $x = 0$.
(4) Find three equations in the three unknowns x, y, z that must be satisfied if $x \neq 0$ at the solution.
(5) Show that $x = 1, y = 1, z = 4$ satisfies these equations.
(6) Write out the bordered Hessian and check the second order conditions for the solution in (5).
(7) Use the sensitivity analysis to estimate the maximum value of $f(x, y, z)$ on the constraint set $x^2 + y^2 + z \leq 6.2, x \geq 0, y \geq 0$ and $z \geq 0$.

Solution : (1) The Lagrange form of this problem is:

$$L = xyz + z - \lambda_1(x^2 + y^2 + z - 6) + \lambda_2 x + \lambda_3 y + \lambda_4 z,$$

the complete set of first order conditions is:

$$(i)\frac{\partial L}{\partial x} = yz - 2\lambda_1 x + \lambda_2 = 0, (ii)\frac{\partial L}{\partial y} = xz - 2\lambda_1 y + \lambda_3 = 0, (iii)\frac{\partial L}{\partial z} = xy + 1 - \lambda_1 + \lambda_4 = 0,$$

$$(iv)\lambda_1(x^2+y^2+z-6) = 0, (v)\lambda_2 x = 0, (vi)\lambda_3 y = 0, (vii)\lambda_4 z = 0, (viii)x^2+y^2+z \leq 6,$$

$$(viv)x \geq 0, y \geq 0, z \geq 0, (vv)\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0.$$

(2) Rewrite (iii) to get $\lambda_1 = xy + 1 + \lambda_4$, from (viv) and (vv), x, y and λ_4 are non-negative, so $\lambda_1 \geq 1 > 0$, from (iv), we must have $x^2 + y^2 + z = 6$, *i.e.*, the constraint $x^2 + y^2 + z \leq 6$ is binding at any solution.

(3)If $x = 0$, then from (i), $yz + \lambda_2 = 0$ and the nonnegativity of y, z and λ_2, we must have $\lambda_2 = 0$ and $yz = 0$, so either $y = 0$ or $z = 0$. If $z = 0$, then from $y^2 + z = 6$, we get that $y = \sqrt{6}$, from (vi), $\lambda_3 = 0$, from (ii), we get that $\lambda_1 = 0$ which contradicts with our conclusion in (2) that $\lambda_1 > 0$. So, we must have $y = 0$, and from $y^2 + z = 6$, we get $z = 6$. From (ii), $\lambda_3 = 0$ and from $(vii), \lambda_4 = 0$, then from $(iii), \lambda_1 = 1$. Therefore, a solution of the first order conditions that include $x = 0$ is

$$(x, y, z, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, 6, 1, 0, 0, 0).$$

(4) If $x \neq 0$, then from (v), $\lambda_2 = 0$, and from (i), we get $yz = 2\lambda_1 x > 0$ and from the nonnegativity of y, z, we must have $y > 0$ and $z > 0$. From (vi) and (vii), we get $\lambda_3 = 0$ and $\lambda_4 = 0$. Therefore, $(i), (iv), (iii)$ and (iv) become

$$(i)'yz = 2\lambda_1 x, (ii)'xz = 2\lambda_1 y, (iii)'xy + 1 = \lambda_1, (iv)'x^2 + y^2 + z = 6.$$

Substituting $(iii)'$ into $(i)'$ and $(ii)'$, we get three equations in the three unknowns x, y, z as follows:

$$yz = 2(xy + 1)x, \quad xz = 2(xy + 1)y, \quad x^2 + y^2 + z = 6$$

(5) It is straightforward to check that $x = 1, y = 1, z = 4$ satisfies the last three equations in (4). And $\lambda_1 = 2, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$, so only the first constraint is binding.

(6) The bordered Hessian is:

$$H = \begin{pmatrix} 0 & 2x & 2y & 1 \\ 2x & -2\lambda_1 & z & y \\ 2y & z & -2\lambda_1 & x \\ 1 & y & x & 0 \end{pmatrix}$$

at $(x, y, z, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 1, 4, 2, 0, 0, 0)$,

$$H = \begin{pmatrix} 0 & 2 & 2 & 1 \\ 2 & -4 & 4 & 1 \\ 2 & 4 & -4 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

since we have three variables and one constraint is binding, so we need to check the last two leading principal minor of H. It is not hard to get that the third order leading principal minor is 64, and the last leading principal minor is -64. So, the sign of the last leading principal minor is the same as $(-1)^3$ (note: $n = 3$) and the last two leading principal minors alternate in sign. So, the solution in (5) is a maximum.

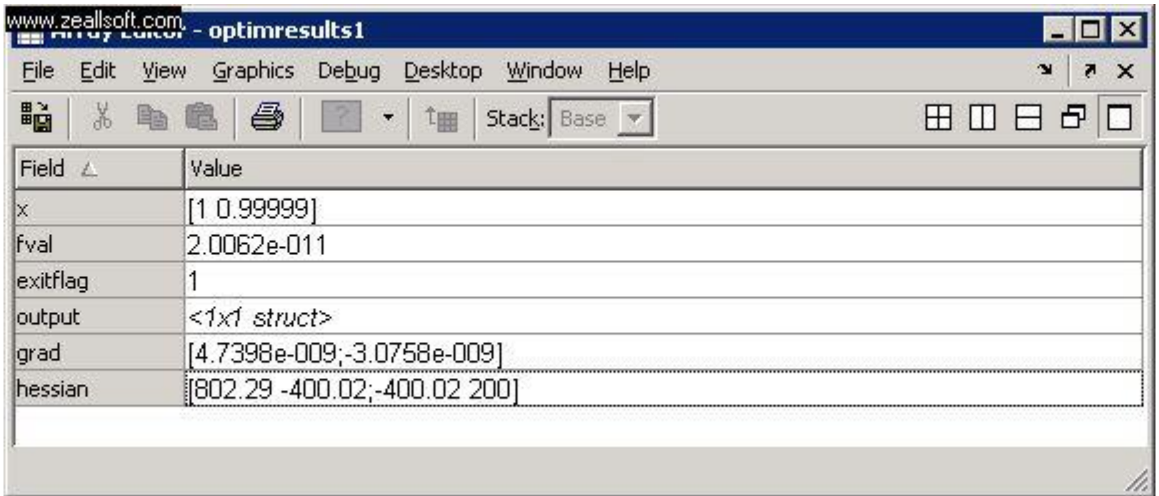
(7) From (3) and (5), the maximum with the constraint set $x^2+y^2+z \leq 6, x \geq 0, y \geq 0$ and $z \geq 0$ is $f(x^*, y^*, z^*) = 8$, and $\lambda_1 = 2$, using sensitivity analysis, when the constraint set becomes $x^2 + y^2 + z \leq 6.2, x \geq 0, y \geq 0$ and $z \geq 0$, the maximum value increases $0.2 \times 2 = 0.4$ to $8 + 0.4 = 8.4$.

Problem 2.

m file of the objective function:

```
function f=objfunp1(x)
f=100*(x(2)-x(1)^2)^2+(1-x(1))^2;
```

Results:



Field	Value
x	[1 0.99999]
fval	2.0062e-011
exitflag	1
output	<1x1 struct>
grad	[4.7398e-009;-3.0758e-009]
hessian	[802.29 -400.02;-400.02 200]

Problem 3.

(1) Fomulate this problem as linear programming problem as follows:

$$\text{maximize} \quad 90x_1 + 150x_2$$ } **Maximize profit**

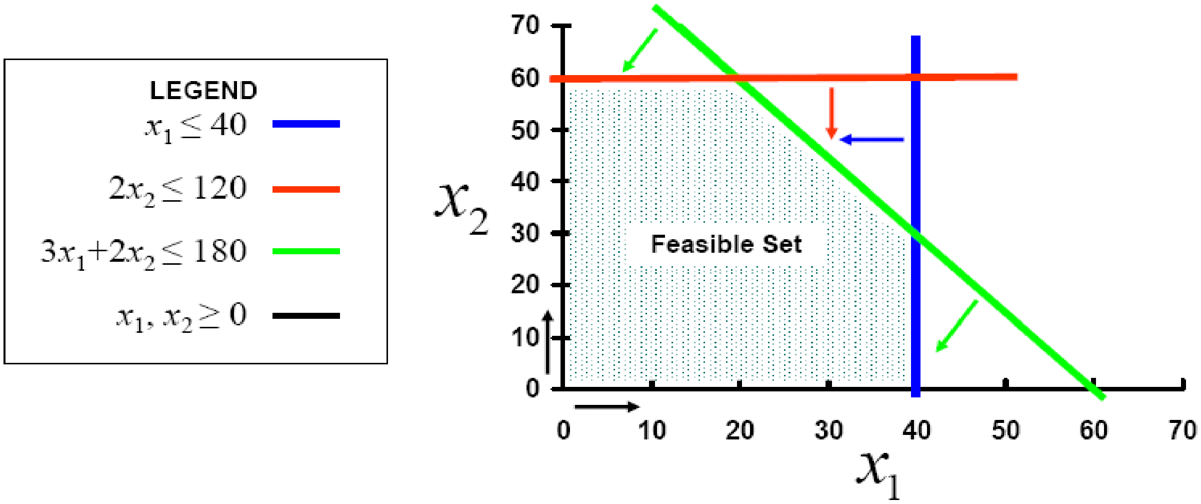
$$\text{subject to:} \quad x_1 \le 40$$ } **Standard compressor capacity constraint**

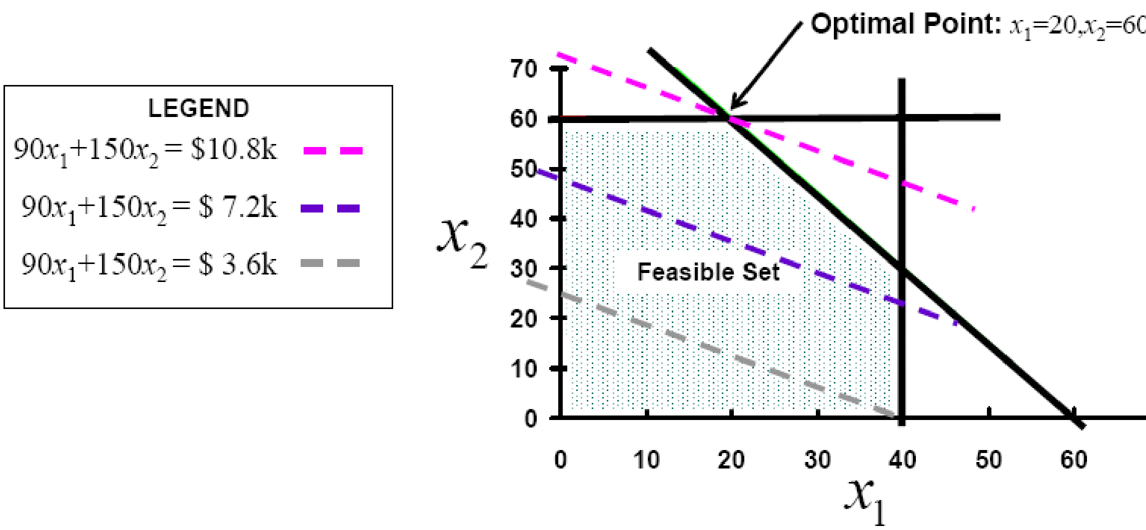
$$2x_2 \le 120$$ } **Hi-pressure compressor capacity constraint**

$$3x_1 + 2x_2 \le 180$$ } **PEM capacity constraint**

$$x_1 \ge 0$$
$$x_2 \ge 0$$ } **The quantities produced must be non-negative**

(2) Solve this linear programming problem graphically

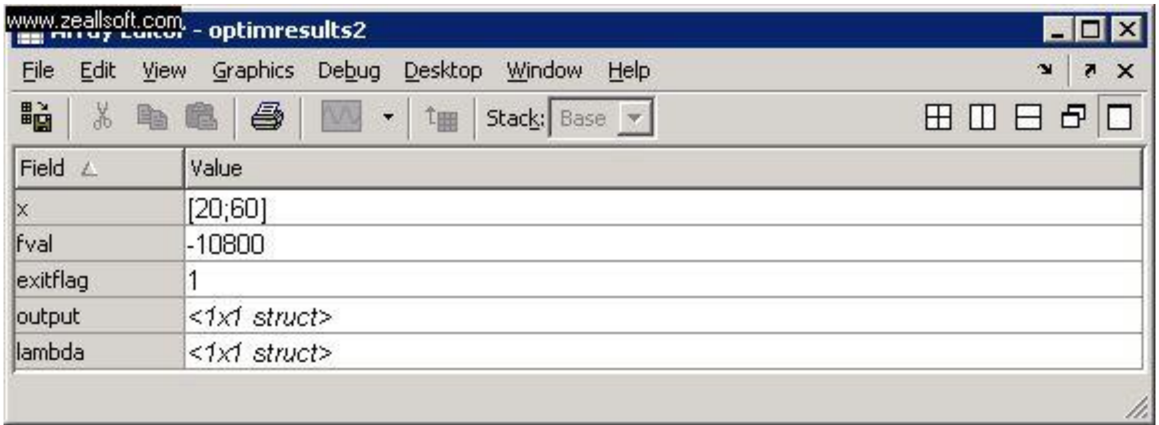




(3) Solve this linear programming in Matlab

```
f=[-90; -150]
A=[1 0; 0 1; 3 2]
b=[40; 60; 180]
lower=[0;0]
```

Result:



Problem 4.

(1) Formulate this problem as a quadratic programming problem.

$$\text{minimize} \quad \sum_{i=1}^{3}\sum_{i=1}^{3} c_{i,j}x_i x_j$$

$$\text{subject to:} \quad 1.08 \le m_1x_1 + m_2x_2 + m_3x_3$$

$$x_1 + x_2 + x_3 = 1$$

$$x_i \ge 0, i = 1,2,3$$

(2) H=2*[0.0163 -0.0137 -0.0020; -0.0137 0.0239 -0.0007; -0.0020 -0.0007 0.0026]
f=[0;0;0]
A=[-1.05 -1.10 -1.07; -1 0 0; 0 -1 0; 0 0 -1]
b=[-1.08; 0; 0; 0]
Aeq=[1 1 1]
beq=1

Result: