

FINANCIAL OPTIMIZATION

Lecture 2: Linear Programming

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A General Linear Program (LP)

Choose x to

minimize $c^\top x$

subject to $(\forall i \in \mathcal{E}) a_i^\top x = b_i$, and

$(\forall i \in \mathcal{I}) a_i^\top x \geq b_i$.

- continuous choice variables x_1, \dots, x_n
- linear objective $c^\top x$
- linear (or affine) equality constraints $a_i^\top x = b_i$
- linear (or affine) inequality constraints $a_i^\top x \geq b_i$
- $\mathcal{E} \cap \mathcal{I} = \emptyset$
- value $c^\top x$ for optimal x

Why Linear Programming

- We can solve even large problems quickly
- Many practical applications
- Can approximate nonlinear convex problems
- Important historically and useful for conceptual understanding

Theoretical Properties of LPs

- for $k > 0$, multiplying c by k does not change the solution
- for $k > 0$, multiplying both a_i and b_i by k does not change the solution
- for $k > 0$, replacing x everywhere by x/k multiplies the solution by k

These theoretical properties are not always respected by numerical optimizers!

Linear Program: Standard Form

Choose $x \geq 0$ to
minimize $c^\top x$
subject to $Ax = b$

convention: $x \geq 0 \Leftrightarrow (\forall i) x_i \geq 0$

This standard form is without loss of generality:

- add a *slack variable* to convert an inequality into an equality
 - ◇ $a_i^\top x \geq b_i$ is equivalent to $a_i^\top x - z_i = b_i$ for $z_i \geq 0$
- replace an unconstrained variable by the difference of two nonnegative variables
 - ◇ replace unrestricted x_j by $x_j - z_j$ where $x_j \geq 0$ and $z_j \geq 0$

Example: Converting to standard form

Convert the following LP to standard form:

Choose $x = (x_1, x_2)$ to
minimize x_2 , subject to

$$x_1 + x_2 \geq 1$$

$$x_1 - x_2 \leq 0$$

Note: x_1 and x_2 are unrestricted in sign.

In-class exercise: converting to standard form

Convert the following LP to standard form:

Choose $x = (x_1, x_2, x_3)$ to
minimize $4x_1 + x_2 - x_3$, subject to

$$x_1 + 3x_3 \leq 6$$

$$3x_1 + x_2 + 3x_3 \geq 9$$

$$x_1 \geq 0, x_2 \geq 0$$

Note: x_3 is unrestricted in sign.

Drawing Pictures: LP

- may be infeasible
- may be unbounded
- solution may not be unique
- if there is a solution, there is a solution at the boundary
- Simplex Algorithm

Pension Fund Application: Asset-Liability Management

This is an application (simplified a lot for classroom use) that has come up in my consulting. A defined-benefit (DB) pension plan has future cash flows (subject only to actuarial risk) that it will have to make in the future based on vesting of current participants. We want to ask what is the cheapest way to generate the cash needed to fund the future liability (and therefore *immunize* the plan against risk in interest rate and other market movements) using Treasury securities. This might be the fund's actual strategy, or it could be an exercise required by the actuaries to compute the value today of the fund's vested liability for the purpose of determining funding status.

Pension Fund Application: Practical Issues

We will need to determine the holdings of Treasury securities, which must be held in nonnegative amount because only the government can issue Treasuries. For each bond, we need to know the cash flows, starting with payment of the all-in purchase price (including the quoted price less any transaction cost and plus any accrued interest). Usually, we put cash flows into time *buckets*, say one per year or two per year, and we have some way of moving the cash flows to the middle of the time bucket (say by recomputing the price using the same yield). If this is an actuarial exercise, we probably have other adjustments, e.g. adding the spread of BAA bonds over Treasuries or maybe we exclude the cost. In most pension exercises, the pension liabilities go beyond the longest Treasury bond, so something must be done about that as well. Our analysis will abstract from these issues and assume our data has already been cleaned to apply for equally-spaced time periods.

Pension Fund Application: Data

We assume that the liability we want to fund is (\$millions)

year 0	year 1	year 2	year 3	year 4	year 5	year 6	year 7	year 8
0	70	72	75	78	82	83	80	77

and the all-in cash flows for the bonds we are using are

	year 0	year 1	year 2	year 3	year 4	year 5	year 6	year 7	year 8
-100.000538377201	103.2	0	0	0	0	0	0	0	0
-100.193067275656	103.4	0	0	0	0	0	0	0	0
-99.6249162430376	3.2	103.2	0	0	0	0	0	0	0
-100.005342595770	3.4	103.4	0	0	0	0	0	0	0
-97.7828297163998	2.8	2.8	102.8	0	0	0	0	0	0
-98.626073410805	3.1	3.1	103.1	0	0	0	0	0	0
-96.771560223781	2.9	2.9	2.9	102.9	0	0	0	0	0
-97.5017227299087	3.1	3.1	3.1	103.1	0	0	0	0	0
-96.546843528248	3.2	3.2	3.2	3.2	103.2	0	0	0	0
-97.4503816435043	3.4	3.4	3.4	3.4	103.4	0	0	0	0
-93.3988985303085	2.9	2.9	2.9	2.9	2.9	102.9	0	0	0
-94.4595953389	3.1	3.1	3.1	3.1	3.1	103.1	0	0	0
-91.4031129930142	2.9	2.9	2.9	2.9	2.9	2.9	102.9	0	0
-92.6025805868467	3.1	3.1	3.1	3.1	3.1	3.1	103.1	0	0
-91.9071119217054	3.3	3.3	3.3	3.3	3.3	3.3	3.3	103.3	0
-93.2590270335428	3.5	3.5	3.5	3.5	3.5	3.5	3.5	103.5	0

Pension Fund Application: LP problem

Given L and B , choose nonnegative x to minimize xP , subject to $xB \geq L$.

- x : 16×1 row vector of bond quantities, the choice variable
- L : 8×1 row vector of liability cash flows for years 1, ..., 8
- P : 1×16 column vector of bond prices, equals -cash flow at year 0
- B : 16×8 matrix of bond cash flows for periods 1, ..., 8

Using Solver in Excel

Solver is not state-of-the-art and can be unreliable, but it serves us well for the course. There are many professional optimization packages available; if I knew which one you would use I would use that one instead. In any case, writing optimization software is partly an art and different packages perform better on different problems.

Some tips:

- To use Solver, you may have to install it first as an add-on.
- For LPs, use "Assume Linear Model" in Options
- Make all variables nonnegative by using "Assume Non-Negative" in Options or enter constraints separately
- Enter constraints using ranges, for example all inequality constraints or all equality constraints at once.
- The new Excel for Mac seems not to include Solver, and Solver seems to have serious problems in the old version. In case of trouble, use the lab machines.

Pension Fund Application: Solution from Solver

bond weights:

(0, 0.5314, 0, 0.5695, 0.6188, 0, 0, 0.6661, 0.7268, 0, 0.7600, 0, 0, 0.7521, 0.7454, 0)

maturities:

(1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8)

shadow prices of cf constraints:

(0.9690, 0.9353, 0.8993, 0.8614, 0.8219, 0.7812, 0.7398, 0.6978)

Cash Management Application (book page 41)

A company must meet the following cash flows over the coming months:

Month	Jan	Feb	Mar	Apr	May	Jun
Net cash flow (\$1,000s)	-150	-100	200	-200	50	300

The following borrowing and lending are available:

- a line of credit, maximum \$100K, interest 1% per month
- in any one of the first three months, issue 90-day paper with total interest of 2% over three months
- invest short-term at 0.3% per month

Obviously, this example is very simple for classroom purposes, but for practice it could be very detailed because linear programming software can handle big problems. This is a standard application, and I included a similar problem in my book *Personal Computing for Managers Using Lotus 1-2-3* many years ago.

Linear Program Representation

Decision variables:

- x_i : amount drawn on the line of credit in month $i = 1, 2, 3, 4, 5$
- y_i : amount of commercial paper to issue in month $i = 1, 2, 3$
- z_i : excess funds to invest in month $i = 1, 2, 3, 4, 5$
- v : the net amount of cash remaining at the end of June

Choose $x_1, \dots, x_5, y_1, \dots, y_3, z_1, \dots, z_5$, and v to maximize v , subject to:

$$x_1 + y_1 - z_1 = 150$$

$$x_2 + y_2 - z_2 - 1.01x_1 + 1.003z_1 = 100$$

$$x_3 + y_3 - z_3 - 1.01x_2 + 1.003z_2 = -200$$

$$x_4 - z_4 - 1.01x_3 + 1.003z_3 - 1.02y_1 = 200$$

$$x_5 - z_5 - 1.01x_4 + 1.003z_4 - 1.02y_2 = -50$$

$$-1.01x_5 + 1.003z_5 - 1.02y_3 - v = -300$$

$$x_1, \dots, x_5 \leq 100$$

$$x_1, \dots, x_5, y_1, \dots, y_3, z_1, \dots, z_5 \geq 0$$

Cash Management: Solution from Solver

value = 92.4969

$(x_1, \dots, x_5) = (0, 50.9804, 0, 0, 0)$

$(y_1, \dots, y_3) = (150, 49.0196, 203.434)$

$(z_1, \dots, z_5) = (0, 0, 351.944, 0, 0)$

Lagrange Multipliers:

monthly cf constraints:

Jan	Feb	Mar	Apr	May	Jun
-1.0373	-1.0302	-1.0200	-1.0169	-1.0100	-1.0000

credit line limit:

Jan	Feb	Mar	Apr	May
0	0	0	0	0

Comparative Statics (Sensitivity Analysis)

What is the change of the value or the solution in response to changing one of the parameters? This is so-called “comparative statics” analysis (in economics or finance) or “sensitivity analysis” (in OR). While it is possible to get the answer to these sorts of questions using a “brute force” approach of solving the problem again and again using different parameter values, it can be easier and more illuminating to look at the Lagrange multipliers at the solution.¹ To be specific, the change in value of the problem when we relax a constraint is approximately the Lagrange multiplier times the amount the constraint is relaxed. (If the analysis is very important for your work, I suggest doing it several ways, and preferably with different software, to minimize the possibility for mistakes.)

¹The idea of using the first- and second-order necessary conditions to obtain signs and magnitudes for comparative statics was proposed and explained with many examples by Paul Samuelson, 1947, *Foundations of Economic Analysis*, Harvard University Press.

Comparative Statics in the Cash Management Problem

As a very simple example, the constraints from the size of the line of credit are not binding and their multipliers are all zero. Therefore, relaxing these constraints by obtaining a larger line is not useful.

As a richer example, having a smaller cash flow to deal with in January is worth 1.0373 in June (when the value is measured) per unit of reduction. This means that it would be useful to obtain 5-month trade credit at 3.5% but not at 4%. The report in Solver also tells us that this credit would be valued the same for a decrease of up to 150, i.e., for the whole amount.

A different element on the report that is useful is the “reduced cost” variable that tells how much it would damage the objective function per unit of increase in the variable that is optimally zero. For example, z_5 has reduced cost of .007, so it would have to be possible to lend money at more than .007 plus the current rate $.003 = 1\%$ in total to make it attractive. A rate more than that would imply an arb with the short-term borrowing. For z_2 , there is the same result but the reduced cost is .00714, which is the *future value* of .007.

Summary

- Linear Programming is a powerful tool for problems with enough structure.
- Fast solutions are available for even very large problems.
- Reduce to standard form using slack variables and difference of two variables.
- Pension asset-liability management is only one of many possible cashflow matching problems.
- The cash management problem makes use of borrowing and lending to meet cash flows.
- Comparative statics can help us to anticipate the impact of inaccurate inputs or changing market conditions.