Supplemental notes: Kuhn-Tucker first-order conditions
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Minimization problem (like in the slides):

Choose $x \in \Re^N$ to
minimize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \geq 0$.

$x = (x_1, ..., x_N)$ is a vector of *choice variables.*
$f(x)$ is the scalar-valued *objective function.*
$g_i(x) = 0$, $i \in \mathcal{E}$ are *equality constraints.*
$g_i(x) \geq 0$, $i \in \mathcal{I}$ are *inequality constraints.*
$\mathcal{E} \bigcap \mathcal{I} = \emptyset$

Kuhn-Tucker conditions:

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$

The feasible solution x^* is called *regular* if the set $\{\nabla g_i(x^*) | g_i(x^*) = 0\}$ is a linearly independent set. In particular, an interior solution is always regular.

If x^* is regular and f and the g_is are differentiable, the Kuhn-Tucker conditions are necessary for feasible x^* to be optimal. If the optimization problem is convex, then the Kuhn-Tucker conditions are sufficient for an optimum.

Maximization problem:

Choose $x \in \Re^N$ to
maximize $f(x)$
subject to $(\forall i \in \mathcal{E}) g_i(x) = 0$, and
$(\forall i \in \mathcal{I}) g_i(x) \leq 0$.

$x = (x_1, ..., x_N)$ is a vector of *choice variables.*
$f(x)$ is the scalar-valued *objective function.*
$g_i(x) = 0$, $i \in \mathcal{E}$ are *equality constraints.*
$g_i(x) \leq 0$, $i \in \mathcal{I}$ are *inequality constraints.*
$\mathcal{E} \bigcap \mathcal{I} = \emptyset$

Kuhn-Tucker conditions:

$\nabla f(x^*) = \sum_{i \in \mathcal{E} \bigcup \mathcal{I}} \lambda_i \nabla g_i(x^*)$
$(\forall i \in \mathcal{I}) \lambda_i \geq 0$
$\lambda_i g_i(x^*) = 0$

(same theorems as on the previous page)

example, Second Model in-class exercise from Lecture 1

Given $p_n > 0$ and $\pi_n > 0$ for $n = 1, ..., N$, and $W_0 > 0$,
choose $x = (x_1, ..., x_N) \in \Re^N$ to
maximize $\sum_{n=1}^N \pi_n x_n$
subject to $\sum_{n=1}^N p_n x_n = W_0$ and
$(\forall n) x_n \geq 0$

$\frac{p_1}{\pi_1} < \frac{p_2}{\pi_2} < ... < \frac{p_N}{\pi_N}$
(states ordered from cheapest to most expensive)

$\nabla f = (\pi_1, ..., \pi_N)$
$\mathcal{E} = \{0\}$, $g_0(x) = \sum_{n=1}^N p_n x_n - W_0$
$\nabla g_0 = (p_1, p_2, ..., p_N)$
$\mathcal{I} = \{1, 2, ..., N\}$, for $n > 0$, $g_n(x) = -x_n$ and $\nabla g_n = (0, ..., 0, -1, 0, ...0)$ with the -1 in the nth coordinant

(Note: LP $\Rightarrow$ gradients do not vary with x.)

Kuhn-Tucker conditions:
$\nabla f = \sum_{n=0}^N \lambda_n \nabla g_n$
for $n = 1, ..., N$, $\lambda_n \geq 0$ and $x_n \lambda_n = 0$

For $n = 1, ..., N$, $\pi_n = \lambda_0 p_n - \lambda_n$ or $\lambda_0 = \pi_n/p_n + \lambda_n/p_n$. Because $\lambda_n/p_n \geq 0$, $\lambda_0 \geq \max(\pi_n/p_n) = \pi_1/p_1$. However, we cannot have $\lambda_0 > \max \pi_n/p_n$ because then complementary slackness would imply all x_n are 0, which would not satisfy the budget constraint. Therefore, we have $\lambda_0 = \pi_1/p_1$ and $\lambda_n = ((\pi_1/p_1) - (\pi_n/p_n))p_n$. This expression for λ_n is positive for $n = 2, ..., N$ (implying that $x_n = 0$ for $n = 2, ..., N$) and zero for $n = 1$. Using the budget constraint to compute $x_1 = W_0/p_1$, we have the unique solution of the Kuhn-Tucker conditions:

$\lambda_0 = \pi_1/p_1$
For $n = 2, ..., N$, $x_n = 0$ and $\lambda_n = ((\pi_1/p_1) - (\pi_n/p_n))p_n > 0$
$x_1 = W_0/p_1$ and $\lambda_1 = 0$

It is easy to verify that this is a feasible solution satisfying the Kuhn-Tucker conditions in a convex optimization. Therefore x is optimal.