Problem Set 4 Answers: Eigenvalues, eigenvectors, and regime-switching models
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1. Consider the matrix

\[ D = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \]

A. Compute the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( A \).

\[
\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 2 \\ -1 & 3 - \lambda \end{pmatrix} \\
= -\lambda(3-\lambda) - 2(-1) \\
= 2 - 3\lambda + \lambda^2 \\
= (\lambda - 2)(\lambda - 1).
\]

Therefore, the eigenvalues are \( \lambda_1 = 2 \) and \( \lambda_2 = 1 \). (The ordering is arbitrary, so saying \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \) would also be correct.)

B. Compute corresponding eigenvectors.

for \( \lambda_1 = 2 \), we have \((D - \lambda_1 I)x = 0\) or

\[
\begin{pmatrix} 0 - 2 & 2 \\ -1 & 3 - 2 \end{pmatrix} x = 0.
\]

The first row tells us that \(-2x_1 + 2x_2 = 0\) or \(x_1 = x_2\) (and the second row tells us the same). Arbitrarily setting \(x_2 = 1\) (which corresponds to choice of scaling), we have that the first eigenvector can be taken to be

\[
x^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

We can confirm this by checking the eigenvalue equation \( Dx = \lambda x \):

\[
\begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \times 1 + 2 \times 1 \\ -1 \times 1 + 3 \times 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
For the second eigenvalue $\lambda_2 = 1$, we have $(D - \lambda_2 I)x = 0$ or
\[
\begin{pmatrix}
0 & 2 \\
-1 & 3
\end{pmatrix}
x = 0.
\]
The first row tells us that $-x_1 + 2x_2 = 0$ or $x_1 = 2x_2$ (and the second row tells us the same). Arbitrarily setting $x_2 = 1$ (which corresponds to choice of scaling), we have that the second eigenvector can be taken to be
\[
x^1 = \begin{pmatrix}
2 \\
1
\end{pmatrix}.
\]
We can confirm this by checking the eigenvalue equation $ Dx = \lambda x$:
\[
\begin{pmatrix}
0 & 2 \\
-1 & 3
\end{pmatrix}
\begin{pmatrix}
2 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \times 2 + 2 \times 1 \\
-1 \times 2 + 3 \times 1
\end{pmatrix} = \begin{pmatrix}
2 \\
1
\end{pmatrix} = 1 \begin{pmatrix}
2 \\
1
\end{pmatrix}.
\]

C. Let $x_0 = (3, 2)^T$. Write $x_0$ as a linear combination of the eigenvectors.

Let $x_0 = c_1 x^1 + c_2 x^2$. Equating the transpose of each side we have $c_1 (1, 1) + c_2 (2, 1) = (3, 2)$, or
\[
c_1 + 2c_2 = 3 \\
c_1 + c_2 = 2
\]
Taking the difference of the two equations, we have that $c_2 = 1$ and therefore from either equation we have $c_1 = 1$. So, $x_0 = x^1 + x^2$.

D. Use the eigenvalues and eigenvectors to compute $A^5x_0$.

\[
A^5x_0 = A^5(x^1 + x^2) = A^5x^1 + A^5x^2 = \lambda_1^5 x^1 + \lambda_2^5 x^2 = 32x^1 + x^2 = \begin{pmatrix}
34 \\
33
\end{pmatrix}.
\]

3. Consider a model with three economic scenarios: (1) healthy economy, (2) recession, and (3) depression. These states are assumed to follow a Markov
switching model in continuous time. From a healthy economy, the economy has a probability per unit time of .05 of moving to a recession but cannot move directly to a depression. From a recession, the economy has a probability per unit time of .03 of moving to a healthy economy and a probability per unit time of .02 of moving to a depression. From a depression, the economy has a probability per unit time of .05 of moving to a recession but cannot move directly to a healthy economy.

A. Let \( \pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))^T \) be the vector of the probabilities of the three states at a future time \( t \) given the information now. Write down a first-order vector ODE satisfied by \( \pi(t) \).

\[
\pi'(t) = A\pi(t)
\]

where

\[
A = \begin{pmatrix}
-0.05 & 0.03 & 0 \\
0.05 & -0.05 & 0.05 \\
0 & 0.02 & -0.05
\end{pmatrix}
\]

B. Find the general solution of the vector ODE given in part A.

First, find the eigenvalues (computing the determinant by expanding around the first column):

\[
0 = \det(A - \lambda I) = \det \begin{pmatrix}
-0.05 - \lambda & 0.03 & 0 \\
0.05 & -0.05 - \lambda & 0.05 \\
0 & 0.02 & -0.05 - \lambda
\end{pmatrix}
\]

\[
= (-0.05 - \lambda)((-0.05 - \lambda)^2 - 0.05 \times 0.02) - 0.05(0.03)(-0.05 - \lambda)
\]

\[
= (-0.05 - \lambda)(\lambda^2 + .1\lambda + .0025 - .0010 - .0015)
\]

\[
= - (\lambda + 0.05)\lambda(\lambda + 1)
\]

Eigenvalues are \( \lambda = 0, -0.05, \) and \(-0.10 \). For the associated eigenvectors, we find for each \( \lambda \) a solution of \( (A - \lambda I)q = 0 \). For \( \lambda = 0 \), we have

\[
\begin{pmatrix}
-0.05 & 0.03 & 0 \\
0.05 & -0.05 & 0.05 \\
0 & 0.02 & -0.05
\end{pmatrix} q = 0.
\]
Starting with \( q_3 = 1 \), the last equation (last row) implies \( q_2 = 5/2 \) and the first equation implies \( q_1 = 3/2 \). So, \((3/2, 5/2, 1)\) is an eigenvector corresponding to the eigen value \( \lambda = 0 \).

For \( \lambda = -0.05 \), we have

\[
\begin{pmatrix}
0 & 0.03 & 0 \\
0.05 & 0 & 0.05 \\
0 & 0.02 & 0
\end{pmatrix}
q = 0.
\]

Starting with \( q_3 = 1 \), the middle equation implies \( q_1 = -1 \) and the first and third equations together imply \( q_2 = 0 \). So, \((-1, 0, 1)\) is an eigenvector corresponding to the eigen value \( \lambda = -0.05 \).

For \( \lambda = -0.10 \), we have

\[
\begin{pmatrix}
0.05 & 0.03 & 0 \\
0.05 & 0.05 & 0.05 \\
0 & 0.02 & 0.05
\end{pmatrix}
q = 0.
\]

Starting with \( q_3 = 1 \), the last equation (last row) implies \( q_2 = -5/2 \) and the first equation implies \( q_1 = 3/2 \). So, \((3/2, -5/2, 1)\) is an eigenvector corresponding to the eigen value \( \lambda = -0.10 \).

Since all the eigenvalues are distinct, the homogeneous solution is

\[
\pi(t) = K_1 \begin{pmatrix} 3/2 \\ 5/2 \\ 1 \end{pmatrix} + K_2 e^{-0.05t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + K_3 e^{-1t} \begin{pmatrix} 3/2 \\ -5/2 \\ 1 \end{pmatrix}.
\]

Since our differential equation is homogeneous, this is also the general solution.

C. Find the solution of the ODE that satisfies the initial condition that we are in a recession at time \( t = 0 \).

\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} = K_1 \begin{pmatrix} 3/2 \\ 5/2 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + K_3 \begin{pmatrix} 3/2 \\ -5/2 \\ 1 \end{pmatrix}.
\]

From the first and third equations, we can see that \( K_2 = 0 \). Then from the first or the third equation we can see that \( K_1 = -K_3 \). Plugging this into the
second equation we can see that $K_1 = 1/5$ and $K_3 = -1/5$. So we have the solution

$$
\pi(t) = \begin{pmatrix} 0.3 \\ 0.5 \\ 0.2 \end{pmatrix} + e^{-0.1t} \begin{pmatrix} -0.3 \\ 0.5 \\ -0.2 \end{pmatrix}
$$

D. We have a possible investment project that requires an initial investment of $100,000. The project pays a cash flow $c_t$ of $7,000/year when the economy is healthy, $1,000/year in a recession, and $0/year in a depression. If the interest rate is 2%, is the net present value

$$
\int_{t=0}^{\infty} e^{-rt} E[c_t] dt - 100,000
$$

of the cash flows positive?

$$
PV = \int_{t=0}^{\infty} e^{-rt} E[c_t] dt \\
= \int_{t=0}^{\infty} e^{-rt} (7000, 1000, 0) \pi(t) dt \\
= 1000 \int_{t=0}^{\infty} e^{-0.02t} (7 \times .3(1 - e^{-0.1t}) + 1 \times .5(1 + e^{-0.1t}) + 0 \times .2(1 - e^{-0.1t})) dt \\
= 1000 \int_{t=0}^{\infty} (2.6e^{-0.02t} - 1.6e^{-0.12t}) dt \\
= 1000 \left( \frac{2.6}{0.02} - \frac{1.6}{0.12} \right) \\
= 116,666.67
$$

So yes, the NPV ($= 116,666.67 - 100,000 = 16,666.67$) is positive.