Problem Set 3 Answers: Linear Programming Duality and Fundamental Theorem of Asset Pricing
P. Dybvig

At the start of class next week, submit only problem 3 for grading. For your study, these are answers to the other questions.

1. Dual LPs

A. Write down the dual LP for the following LPs

The first primal LP is:

Choose nonnegative x_1 and x_2 to
minimize $x_1 + 3x_2$, subject to
$x_1 + x_2 \geq 4$.

The first dual LP is:

Choose nonnegative y to
maximize $4y$, subject to
$y \leq 1$ and
$y \leq 3$

The second primal LP is:

Choose nonnegative x_1 and x_2 to
minimize $x_1 - 2x_2$, subject to
$x_1 \geq 1$ and
$x_1 + 2x_2 \geq 2$.

The second dual LP is:

Choose nonnegative y_1 and y_2 to
maximize $y_1 + 2y_2$, subject to
$y_1 + y_2 \leq 1$ and
$2y_2 \leq -2$.

B. For each primal and dual LP, answer the following. Is the LP feasible? Is the LP bounded? If the LP has a solution, what is it?

first primal:
... feasible? yes $x = (4, 4)$ is feasible
... bounded? yes, any direction $\Delta \geq 0$ that does not make the nonnegativity constraints less binding must increase the objective function
... exist a solution? yes, $x = (4, 0)$ is a solution (the other candidate would be $x = (0, 4)$ but the value is larger there (12 vs 4).

first dual:
... feasible? yes $x = 0$ is feasible
... bounded? yes, any direction $\Delta \geq 0$ that does not make the nonnegativity constraints less binding will make the other constraints more binding ($\Delta \not\leq 0$)
... exist a solution? $y = 1$ is a solution

second primal:
... feasible? yes $x = (1, 1)$ is feasible
... bounded? no $\Delta = (1, 1) \geq 0$ reduces the objective and does not make the constraints more binding ($\Delta_1 \geq 0$ and $\Delta_1 + \Delta_2 \geq 0$)
... exist a solution? no (adding $(1, 1)$ to any candidate solution is still feasible and increases value)

second dual:
... feasible? no, since $y_2 \geq 0$, $2y_2 \not\leq -2$
... bounded? yes, any direction $\Delta \geq 0$ that does not make the constraints more binding has $\Delta_1 + \Delta_2 \leq 0$ and $2\Delta_2 \leq 0$ and therefore the sum $(\Delta_1 + \Delta_2) + (2\Delta_2)/2 = \Delta_1 + 2\Delta_2 \leq 0$. Consequently, the objective function cannot be improved: $\Delta_1 + 3\Delta_2 \not> 0$.
... exist a solution? no, since it is not feasible

C. Discuss the results in terms of the duality theorems.

The problems in the first primal and dual pair are bounded and feasible, so they should have solutions with the same value. The value of the primal is $4 + 3 \times 0 = 4$, and the value of the dual is $4 \times 1 = 4$.

Both sets of problems satisfy the result that the primal is feasible if and only if the dual is bounded, and the primal is bounded if and only if the dual is feasible. For the first problem, this is true because both primal and dual are both feasible and bounded. For the second problem, the primal is feasible and the dual is unbounded, while the primal is unbounded and the dual is infeasible.

2. Finding arbitrage: puts

You can buy HAL stock for $59.36/share or go short at $59.32 and you can also buy or sell listed puts all maturing on the same date in August (these are all-in prices based on most recent trades plus an estimate of half the spread plus trading costs):

strike	put ask	put bid
45	1.52	1.48
50	2.77	2.73
55	2.99	2.85
60	5.95	5.85
65	10.65	10.35

In addition, the riskless borrowing rate for the maturity of the options is 1% simple interest and the lending rate is 0.5% simple interest.

A. Set up the state-space tableau for short and long positions in the put options and the underlying stock and riskfree borrowing/lending. Calculate the payoffs based on terminal stock prices 0, 45, 50, 55, 60, 65, and 1000 (all the strike prices plus two extreme values).

The state-space tableau for payoffs at maturity for long positions only is

$$X = \begin{pmatrix} 45 & 50 & 55 & 60 & 65 & 0 & 100 \\ 0 & 5 & 10 & 15 & 20 & 45 & 100 \\ 0 & 0 & 5 & 10 & 15 & 50 & 100 \\ 0 & 0 & 0 & 5 & 10 & 55 & 100 \\ 0 & 0 & 0 & 0 & 5 & 60 & 100 \\ 0 & 0 & 0 & 0 & 0 & 65 & 100 \\ 0 & 0 & 0 & 0 & 0 & 1000 & 100 \end{pmatrix}$$

and the vector of bid prices is $p_b^\top = (1.48, 2.73, 2.85, 5.85, 10.35, 59.32, 99.099)$, and the vector of ask prices is $p_a^\top = (1.52, 2.77, 2.99, 5.95, 10.65, 59.36, 99.50249)$, where we computed $99.099 = 100/1.01$ and $99.50249 = 100/1.005$.

Therefore, the full state-space tableau for cash flows initially and at maturity with short and long positions separately is given by

$$Z = \begin{pmatrix} -p_a^\top & p_b^\top \\ X & -X \end{pmatrix}.$$

For this analysis, assume these are European options.

B. Write down a linear programming problem that searches for an arbitrage given these trading opportunities.

Choose η to
maximize $\mathbf{1}^\top Z\eta$, subject to
$Z\eta \geq 0$
$\mathbf{1}^\top Z\eta \leq 1$.

(Other LPs are possible.)

C. Use Solver to search for an arbitrage opportunity.

In my implementation, I entered the matrix Z in the transpose (with rows and columns reversed), since more colums fit on the screen than rows in Excel. I did not compute $\mathbf{1}^\top Z$ in advance, although that might run faster: this problem is small enough so it doesn't matter. The optimal solution found by solver buys .168067 each of the puts with strike 45 and 55 and sells .336134 of the put with strike 50.

(This solution is not unique.)

D. Examine the optimal solution found by Solver and describe it qualitatively.

The strategy is a butterfly spread and pays off up front and when the stock price ends near 50 but pays zero when the stock price ends below 45 or above 50. In practice, we would have to confirm we can trade at the prices. It is likely that the furthest out-of-the-money option does not trade much and the "most recent" trade's price may not be current.

(If a different solution is found, the discussion would be different. This is not the only good discussion of this solution.