Problem Set 2: Linear Programming
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At the start of class next week, submit only problem 3 for grading.

1. Standard form

Convert the following LPs to standard form. Be sure to explain the relation between the variables in the new and old problems.

Choose $x_1 \geq 0$ and $x_2 \geq 0$ to
maximize $x_1 + x_2$, subject to
$x_1 \leq 6$ and $x_1 + 2x_2 \leq 3$.

Choose x_1 and x_2 to
maximize $2x_1 + x_2$, subject to
$x_2 - x_1 \leq 1$, $x_1 - x_2 \leq 1$, $x_2 + x_1 \geq -1$, and $-x_1 - x_2 \geq -1$.

2. Asset-Liability Application and Lagrange Multiplier

Modify the pension fund example from class (spreadsheet available on the class page) to include an additional bond paying 14 each in years 1 through 8.

A. First, use the Lagrange multipliers to show that if the bond costs 100 initially it is too expensive and will not be held.

B. Verify this by adding the bond with price 100 in solver and checking that the solution is unchanged.

C. Try instead adding the bond with price 90 and verify that the solution changes.

D. Describe in words the original solution and how the solution changes when the new bond is introduced. (This should be like the description you would give a boss who wants a verbal description, not all the quantitative details.)

3. Asset-Liability Application and Lagrange Multiplier

Modify the pension fund example from class (spreadsheet available on the class page) to include 2-year lending at a fixed interest rate in every year.

A. First, use the Lagrange multipliers to show that availablity each period of riskless lending at 4% at a term of two years changes the solution.

B. Verify this by adding riskless 2-year lending at 4% in each year and verify that the solution changes.

C. Describe in words the original solution and how the solution changes when a new lending opportunity is introduced.

D. Try instead adding riskless 2-year lending at 3% and show that the solution does not change.

4. Challenger: approximating a concave function

A concave differentiable function can be written as the minimum of affine functions.[1] and specifically if $u(x)$ is a concave differentiable function defined on an open convex set X, we can write

$$u(x) = \min\{u(\hat{x}) + \nabla u(\hat{x})^\top (x - \hat{x}) | \hat{x} \in X\}$$

or if x is one-dimensional,

$$u(x) = \min\{u(\hat{x}) + u'(\hat{x})(x - \hat{x}) | \hat{x} \in X\}.$$

If we can take a grid of points $G = \{g_1, ..., g_n\}$, we can approximate $u(x) \approx \min\{u(\hat{x}) + u'(\hat{x})(x - \hat{x}) | \hat{x} \in G\}$. Use this approximation to find an LP that approximates the following nonlinear program.

[1] A convex function is the maximum of affine functions. For non-differentiable functions, substitute an element from the subgradient correspondence for the gradient or derivative.

Given W_0,
choose x_1, ... , x_N to
maximize $\sum_{i=1}^{N} \pi_i u(x_i)$, subject to
$\sum_{i=1}^{N} p_i x_i = W_0$.

What special care has to be taken to be sure the approximate problem is unbounded and the solution is nonnegative?