MATHEMATICAL FOUNDATIONS FOR FINANCE

Regime-switching Models

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Regime-switching Models

Regime-switching models say that the economy or part of it sometimes switches at random times to a distinctly different state. Examples of regime-switching in finance include models of credit ratings, volatility of stock prices or interest rates, overall state of the economy (healthy, recession or depression), or in an insurance model the state-of-health or mortality of an individual. Regime-switching models are models with finitely many discrete states. This lecture will consider regime-switching models in continuous time; discrete time models with regime switching are similar to the population growth example we considered earlier.

Often, regime-switching models repesent only part of the analysis. For example, we may have an underlying regime-switching model of volatility and then the stock-price process follows a continuous-time process with continuous state-space whose volatility jumps when the regime changes.

We will use eigenvector-eigenvalue analysis to study continuous-time regime change models. The analysis uses a first-order vector differential equation, and the analysis is very similar to the analysis of the first-order vector difference equation in the population growth example.

Probability Dynamics

Looking forward, the probability distribution of states satisfies a first-order differential equation with constant coefficients. (In some models, this could be more complicated, but this is what we are studying today.) Suppose there are n states, $\theta = 1, 2, ..., n$, and the probability now of being in state θ at some future time t is $\pi_i(t)$. We are thinking that in some small time interval $[t, t + \Delta t]$ there is a probability of order Δt of having a jump to another state and a very small probability of order $(\Delta t)^2$ of having more than one jump. We also take the probability of changing from one state to another is only dependent on the state θ_t at time t and not on the previous history. This says that future dynamics depend on the past through only the current state, and this is call the Markov property. Because we have the Markov property and finitely many states, we can say this is a finite Markov Chain.

Assuming the state dynamics do not vary over time, the probability vector $\pi(t) = (\pi_1(t), \pi_2(t), ..., \pi_n(t))^T$ satisfies the differential equation

$$\pi'(t) = A\pi(t),$$

where A is an $n \times n$-matrix of state-transition rates.

Preservation of Probabilities

Because probabilities must add to one at each point in time, we can derive a restriction on the matrix A. The sum of the probabilities at time t is $\mathbf{1}^T\pi(t)$, which is constant so its time derivative is always 0:

$$0 \equiv (\mathbf{1}^T\pi(t))' = \mathbf{1}^T\pi'(t) = \mathbf{1}^T A\pi(t).$$

Since this must be zero for all nonnegative $\pi(t)$ that add to 1, it must be that $\mathbf{1}^T A = 0$. We can interpret this as saying that all columns of A sum to zero. This means that any probability that comes out of one state must to into another state.

Nonnegativity of Probabilities

Another property of probabilities is that they cannot be negative. To preserve this, we need that $\pi_i'(t) \geq$ whenever $\pi_i(t) = 0$. This implies that the off-diagonal elements of A are nonnegative (positive or zero). For $i \neq j$, if $\pi_j = 1$ (so $\pi_i = 0$ as are all the π_k's for $k \neq i$ or j), then $\pi_i'(t) = A_{ij}$which must therefore be positive. In fact, it is not hard to show that this implies that $\pi_i'(t) \geq$ whenever $\pi_i(t) = 0$.

Since the off-diagonal elements of A are nonnegative and the columns sum to 0, it follows that the on-diagonal elements are nonpositive (negative or zero). This implies another important property, that probabilities can never go above 1. If $p_i(t) = 1$ (so all the other $p_j(t)$'s are zero), then $p_i'(t) = A_{ii} \leq 0$.

An Example: State of the Economy

Let's consider a model with three states of the economy: good times (state 1), recession (state 2), and depression (state 3). In good times, the economy switches to a recession at a probability rate of 0.1 per unit of time. In a recession, the economy switches to good times at a rate of 0.2 per unit time and to depression at a rate of 0.2 per unit time. In a depression, the economy switches to a recession at a rate of 0.4 per unit time. In terms of the dynamics $\pi'(t) = A\pi(t)$, we have that

$$A = \begin{pmatrix} -0.1 & 0.2 & 0 \\ 0.1 & -0.4 & 0.4 \\ 0 & 0.2 & -0.4 \end{pmatrix}$$

We will assume the economy is initially in a recession (state 2), i.e., $\pi(0) = (0, 1, 0)^T$.

Why an Eigenvalue Decomposition of A is Useful

We want to do an eigenvalue-eigenvector decomposition of A. To see why this is useful, let λ be an eigenvalue of A with associated eigenvector q, and let c be any constant. Then $\pi(t) = ce^{\lambda t}q$ is a solution to the equation $\pi'(t) = A\pi(t)$. If we add together three such solutions, we have the general solution to the problem.

The first step is to do the eigenvalue-eigenvector decomposition of A. This is a $3x3$ matrix, so it is a little harder than the examples we have done so far. However, it is a singular matrix so one of the eigenvalues is 0, so it is not so bad.

Eigenvalue Decomposition of A

The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = -.2$, and $\lambda_3 = -.7$, with associated eigenvalues proportional to $q_1 = (4,2,1)^T$, $q_2 = (-2,1,1)^T$, and $q_3 = (1/2,-3/2,1)^T$, respectively.

As a result, the general solution of $\pi'(t) = A\pi(t)$ is given by

$$\begin{aligned}\pi(t) &= c_1 \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{-.2t} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-.7t} \begin{pmatrix} 1/2 \\ -3/2 \\ 1 \end{pmatrix} \\ &= Mc(t)\end{aligned}$$

where

$$M \equiv \begin{pmatrix} 4 & -2 & 1/2 \\ 2 & 1 & -3/2 \\ 1 & 1 & 1 \end{pmatrix}$$

is a matrix whose columns are the eigenvectors and $c(t) \equiv (c_1, c_2 e^{-.2t}, c_3 e^{-.7t})^T$.

Initial Condition

The initial condition is that we start in state 2, i.e. that $\pi(0) = (0, 1, 0)^T$. Then $c = (c_1, c_2, c_3) = c(0)$ solves the equation

$$Mc = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The solution is $c = (1/7, 1/5, -12/35)^T$

Degenerate Case

Consider the solution $\gamma(t)$ of the differential equation $\gamma'(t) = B\gamma(t)$ where

$$B \equiv \begin{pmatrix} 0 & .1 & 0 \\ 0 & -.1 & .1 \\ 0 & 0 & -.1 \end{pmatrix}$$

This is the probability transition assuming there is a one-way transition from state 3 to state 2 to state 1, each at probability $.1$ per unit time. The matrix B has eigenvalue $\lambda_1 = 0$ with eigenvector $q_1 = (1, 0, 0)^T$, but it also has an eigenvalue $\lambda_2 = \lambda_3 = -.1$, a double root, with only one eigenvector $q_2 = (-1, 1, 0)$. So, we have a solution $c_1 q_1 + c_2 e^{-.1t} q_2$, but we should have three free parameters and we only have two.

What to do?

Finding the solution step-by-step

We can write our system of differential equations as

$$\begin{aligned}
\gamma_1'(t) &= .1\gamma_2(t) \\
\gamma_2'(t) &= -.1\gamma_2(t) + .1\gamma_3(t) \\
\gamma_3'(t) &= -.1\gamma_3(t)
\end{aligned}$$

Fortunately, we can solve first for $\gamma_3(t)$ (which did not appear in the eigenvalues).

$$\gamma_3(t) = d_3 e^{-.1t}$$

Next step

Now we can write the remaining system as

$$\begin{aligned}\gamma_1'(t) &= .1\gamma_2(t)\\ \gamma_2'(t) &= -.1\gamma_2(t) + .1d_3e^{-.1t}\end{aligned}$$

Now, this is not degenerate, and we can find the solution the normal way. Or, we can solve a one-dimensional equation for $\gamma_2(t)$ which is

$$\gamma_2(t) = .1d_3te^{-.1t} + d_2e^{-.1t}$$

and then for $\gamma_1(t)$ which is

$$\gamma_1(t) = -.1d_3te^{-.1t} - (d_3 + d_2)e^{-.1t} + d_1$$

Initial Condition

Suppose we have an initial condition that we are in state 3 (the most interesting case). Then $\gamma(0) = (0, 0, 1)^T$ and we have that $d \equiv (d_1, d_2, d_3)^T = (1, 0, 1)^T$ so the solution is

$$\gamma(t) = \begin{pmatrix} 1 - .1te^{-.1t} - e^{-.1t} \\ .1te^{-.1t} \\ e^{-.1t} \end{pmatrix}.$$