Portfolio Turnpikes

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Abstract

Portfolio turnpike theorems show that if preferences at large wealth levels are similar to power utility, then the investment strategy converges to the power utility strategy as the horizon increases. We state and prove two simple and general portfolio turnpike theorems. Unlike existing literature, our main result does not assume independence of returns and depends only on discounting of future cash flows. We also provide a critique of portfolio turnpike results, based on the observations that (1) the time required for convergence is often too large to be relevant, and (2) there is no convergence for consumption withdrawal problems.

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Introduction

Turnpike theorems in finance make a seductive promise: when the horizon is long, we can obtain essentially optimal portfolio weights by solving a relatively simple problem assuming power utility with a shape similar to that of the correct utility funtion at large wealth levels. Although the literature contains a number of these results with different technical variations, the main assumptions that are common in the existing literature are (1) returns are independent over time (and in most papers i.i.d.), and (2) investments can grow over time because the riskless rate is positive. It is the purpose of this paper to provide a critical examination of this literature and provide a new perspective on these results. There are two main contributions in this paper. One is to provide a simple and general turnpike result that helps to put the literature in perspective. The second is to provide numerical examples that indicate whether convergence is fast enough for practical use. Our main findings are (1) it is the growth of the economy as reflected in interest rates or discount bond prices, not independence, that is critical for the results, and (2) convergence is too slow to be of practical interest, provided we assume real rates of interest are small enough to be plausible. We conclude that while portfolio turnpike theorems enhance our intuition and understanding of portfolio problems, they are not particularly useful in practice.

The term "turnpike theorem" has its origins in growth theory. According to Dorfman, Samuelson, and Solow [1958],

> It is, in a sense, the single most effective way for the system to grow, so that if we are planning long-run growth, no matter where we start and where we desire to end up, it will pay in the intermediate stages to get into a growth phase of this kind. It is exactly like a turnpike paralleled by a network of minor roads. There is a fastest route between any two points; and if origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means

> adding a little mileage at either end. The best intermediate capital configuration is one which will grow most rapidly; even if it is not the desired one, it is temporarily optimal.

Based on this analogy, these results are called turnpike theorems, and a significant literature has grown out of this idea. For an agent maximising expected utility of terminal wealth at a distant horizon, portfolio turnpike theorems say that the agent's optimal portfolio is insensitive to properties of the utility function at low wealth levels. Such results always assume a market which is growing indefinitely (as they clearly must); it is then not surprising that the values of the utility at low wealth levels are unimportant, as the agent can always get away from these low levels simply by following the growing market. In particular, portfolio turnpike theorems say that for all utility functions that are similar (in some suitably-defined sense) to a power utility function at large wealth levels, the optimal portfolio strategy spends most of its time over a large horizon following a portfolio strategy similar to the portfolio strategy of the power utility function, a neighborhood of which is the turnpike.

To introduce turnpike theorems without the full weight of the formal model, we provide examples in §1. These examples form the basis of our critique of turnpike results. Our critique looks at two separate problems. First, the optimal path may not lie near the turnpike unless one has an extremely long horizon. Examples with reasonable parameter values suggest that it may take a horizon in excess of 50 or 100 years before the optimal portfolio choice is close to its asymptotic value, even if the utility function is identical above the initial wealth level. The rate of interest, properly interpreted as a real rate of interest, seems to be a critical parameter in determining the rate of convergence. Faster convergence would require us to assume an unreasonably large real rate of interest. The slow convergence suggests that the portfolio turnpike results are of little practical import: using the asymptotically correct strategy may be far from optimal even at the largest horizons likely to be encountered in practice.

The second problem with the turnpike results is that they do not hold for consumption-withdrawal problems. While many investment problems may have overall horizons that are very large or even unbounded (such as the management of a university endowment), these problems involve ongoing withdrawals for consumption, which are assumed away by the structure of the portfolio turnpike models. Portfolio turnpike theorems can be relied on as a useful approximation only when the time until the first consumption withdrawal is very large, and this is unlikely to be encountered in practice.

In §2 we describe the formal model, and state our main result, Theorem 1; this is a comparison theorem which contains many existing results in the literature. We show that if two agents have similar marginal utilities at large consumption levels, they must have nearly the same wealth process and portfolio strategy at early times when the horizon is distant. The intuition for our main result is simple: assuming positive interest rates (or something like that), our portfolio outgrows the low wealth levels for which the two utility functions are significantly different. If all reinvestment were at the riskless rate, it would be obvious that it is the shape of the utility function at large wealth levels that governs the indirect utility function at short horizons. What is more subtle is to see that the states of nature with low optimal consumption at the end, while occuring with positive probability given the presence of risk-taking, are not very significant economically, and have a small influence on initial portfolio choice. From the previous literature, it might seem that this follows from independence of returns and some sort of law of large numbers; our results do not require independence and therefore we conclude that it is discounting alone, not discounting combined with independence, that drives turnpike results.

This main result assumes complete markets in a continuous-time model; the local means and variances of security returns can follow fairly general adapted processes. We impose regularity through existence of moments of the state price density, rather than through specific assumptions about the returns themselves, such as assuming that returns are independent over time or that stock prices are diffusions. We consider the portfolio strategies of two agents, and unlike the literature, we do not assume that either agent necessarily has constant relative risk aversion. Utility functions satisfy a uniform continuity property which certainly holds if the relative risk aversion is bounded above and below.

One result which is not covered by Theorem 1 is that of Huberman and Ross [1983]. Apart from the inessential difference of being stated in discrete time, their result uses a weaker notion of equivalence of utilities (regular variation of marginal utilities, with the same exponent), but it makes a stronger assumption (independence over time) about returns. Under the continuous-time analogues of these assumptions, we prove (Theorem 2) the continuous-time analogue of the Huberman-Ross result. This is the first continuous-time result of this sort, and provides a bridge between the discrete- and continuous-time literatures. One innovation in this result is that we assume much less smoothness on preferences than do Huberman and Ross or the rest of the literature. This is possible because of a result that shows that there is a smooth utility function with a slightly different horizon that has exactly the same portfolio choice and wealth process. This result permits us to apply the results assuming smooth preferences to preferences with kinks.

The literature on portfolio turnpike theorems includes discrete-time models (Mossin (1968), Leland (1972), Hakansson (1974), Huberman and Ross (1983)) and continuous-time models (Cox and Huang (1992), Huang and Zariphopoulou (forthcoming)). All of these papers assume *i.i.d.* returns, with the exception of Huberman and Ross. Huberman and Ross assume returns are independent across periods, with bounded support. We do not assume independence in our main result. In all of the previous literature, it was also assumed that the reference utility function has constant relative risk aversion, which we have not assumed (instead, we assume the weaker uniform continuity condition (27)).

There are different assumptions in the literature regarding how the utility function converges to the reference utility function at large wealth levels. In order to compare our assumption (26) made in § 2 to the previous literature, we need to specialize our model by assuming the reference utility function has constant relative risk aversion. With this specialization, (26) is the same assumption made by Huang and Zariphopoulou and is strictly more general than the assumptions of Mossin, Leland, and Cox and Huang. However, it is less general than that of Huberman and Ross, and apparently simply different from Hakansson's.

The proofs of both Theorem 1 and Theorem 2 are relegated to an appendix, although the text does outline the main ideas in the proofs. Convergence of relative risk aversion implies regular variation, which in turn implies the regularity condition of Theorem 1, which shows that many results in the literature can be read off from our main results. These results and related comparisons are shown in Lemma 1. § 3 closes the paper, and the Appendix contains the proofs.

1 Examples and Critique

The portfolio turnpike results tell us that, given utility functions that are asymptotically similar at large wealth, the portfolio strategies are asymptotically similar at large horizon. This section uses examples to help the reader to develop an intuition for the turnpike results and their limitations.

The major limitations of the turnpike results may be summarized in two critiques: firstly, examples show that, with reasonable parameter values (especially when the real riskless rate is reasonably small), the convergence may be slow (even when the utility function differs from a power function only at levels below the initial wealth, we may not be near convergence even with a horizon as long as 100 years!); and, secondly, we should not expect any turnpike results in consumption withdrawal problems. This second critique is implicit or explicit in many of the turnpike papers, but it is worthy of attention, especially in combination with the other critique: no consumption withdrawal over 50 years seems unusual.

In later sections of the paper, we will specify our formal model more precisely, but for now we will present just enough notation to be able to present the examples without proof. Throughout the paper, we will assume a continuous-time model in which the underlying uncertainty is generated by a standard Wiener process that may be multi-dimensional. For the examples, we will specialize this to fixed coefficients in a world with a one-dimensional Wiener process and a single riskless asset. We will take r to be the fixed riskless rate, μ to be the fixed mean return on the risky asset, and σ to be the fixed standard deviation of return on the risky asset. As is well known, the effective budget constraint in this problem can be written as $W_0 = E[C\xi_T]$, where W_0 is initial wealth, C is consumption, T is the horizon, and $\xi_t = \exp(-(r-\gamma^2/2)t - \gamma Z_t)$ for $\gamma \equiv (\mu - r)/\sigma$. Finally, we choose in this section to remain vague on the definition of when two utility functions are "similar at large wealth levels." Suffice it to say that there are a number of definitions in the literature and that our examples have utility functions that are similar whatever definition we use (although as an inessential matter they may not satisfy regularity assumed by the literature at low wealth levels). Formal definitions are given in § 2.

Example 1

Here, we take utilities

$$u_0(C) \equiv \begin{cases} \frac{C^{1-R}}{1-R} & \text{for } C > 0 \\ -\infty & \text{for } C \leq 0 \end{cases} \tag{1}$$

and

$$u_1(C) \equiv \begin{cases} \frac{(C-K)^{1-R}}{1-R} & \text{for } C > K \\ -\infty & \text{for } C \leq K \end{cases} \tag{2}$$

where K is the translation and $R > 0$ is the shared risk-aversion parameter.[1] It is easy to verify that these two utility functions are similar in the sense of (26) given that the R's are the same.

With u_0 and u_1 defined in this way, the solutions for the two utility functions are closely related for reasons given by Cass and Stiglitz [1970]. By a simple

[1] Formally, when K is negative we want to relax the nonnegative wealth constraint in a way that does not create an arbitrage so negative consumption can be permitted, e.g. by some sort of L^p integrability condition or a looser lower bound on wealth. Addressing this purely technical issue in detail would take us too far afield of our main purpose.

change of variables that converts the problem for one utility function into the other we have that any solution has the property that

$$C_{1T} = K + \frac{W_0 - KE[\xi_T]}{W_0} C_{0T} \tag{3}$$

relates the two consumptions. If we take r to be constant (as we will for the calculations), the portfolio investment needed to achieve K uses only the riskless asset, and the portfolios are related by

$$\theta_{1t;T} = \frac{W_0 - Ke^{-rT}}{W_0} \theta_{0t;T}, \tag{4}$$

since the discount factor is $E[\xi_T] = e^{-rT}$ when r is constant. By the turnpike theorem (or by direct computation), the two portfolio strategies converge as T increases. According to this result, the relative error for agent 1 from using the asymptotic risky asset portfolio $\theta_{0t;T}$ instead of the correct one, defined to be

$$(\theta_{0t;T} - \theta_{1t;T})/\theta_{1t;T},$$

is given by $K/(W_0 e^{rT} - K)$, independent of R and the parameters of the risky asset return processes.

For power utility translated by 50% of initial wealth, Tables 1 and 2 show the percentage error we would make in choosing what is optimal for the power utility (as is asymptotically correct as the horizon increases) instead of what is actually optimal. At reasonable real interest rates (2% or 4%) convergence is probably too slow to make this a useful approximation.

Example 2

The class of examples based on translated power utility is very suggestive that convergence tends to be slow. However, one weakness of this class of

Portfolio error (%): power utility translated 50% W_0							
	years to maturity						
r	1	2	5	10	25	50	100
2%	96.12	92.45	82.62	69.31	43.53	22.54	7.26
4%	92.45	85.72	69.31	50.41	22.54	7.26	0.92
6%	88.99	79.68	58.83	37.82	12.56	2.55	0.12
8%	85.72	74.24	50.41	28.98	7.26	0.92	0.02
10%	82.62	69.31	43.53	22.54	4.28	0.34	0.00

Table 1: The table gives the percentage error from using the asymptotic value instead of the optimum. For example, if it is optimal to invest 50% of wealth in the risky asset, an entry of 10.00 in the table implies the asymptotic rule would give 55% instead. The utility function is constant risk aversion translated by 50% of initial wealth, i.e., $K = W_0/2$. The entries in this table are not sensitive to μ (if not equal to r), σ (if not zero), or R (if positive). The annual (real) riskless rate r and the number T of years to maturity are varied in the table.

Portfolio error (%): power utility translated -50% W_0							
	years to maturity						
r	1	2	5	10	25	50	100
2%	-32.89	-32.45	-31.15	-29.05	-23.27	-15.54	-6.34
4%	-32.45	-31.58	-29.05	-25.10	-15.54	-6.34	-0.91
6%	-32.01	-30.72	-27.03	-21.53	-10.04	-2.43	-0.12
8%	-31.58	-29.88	-25.10	-18.34	-6.34	-0.91	-0.02
10%	-31.15	-29.05	-23.27	-15.54	-3.94	-0.34	0.00

Table 2: The table gives the percentage error from using the asymptotic value instead of the optimum. For example, if it is optimal to invest 50% of wealth in the riskless asset, an entry of -10.00 in the table implies the asymptotic rule would give 45% instead. The utility function is constant risk aversion translated by -50% of initial wealth, i.e., $K = -W_0/2$. The entries in this table are not sensitive to μ (if not equal to r), σ (if not zero), or R (if positive). The annual (real) riskless rate r and number T of years to maturity are varied in the table.

examples is that the entire utility function is changed (at least somewhat) by the translation. To counter this, Example 2 assumes power utility above W_0 and globally minimal utility (corresponding to the limit of infinite risk aversion) below W_0:

$$(5) \quad u_1(C) \equiv \begin{cases} \frac{C^{1-R}}{1-R} & \text{for } C \geq W_0 \\ -\infty & \text{for } C < W_0 \end{cases}$$

The first-order condition for an optimum implies that

$$(6) \quad C = \begin{cases} (\lambda\xi_T)^{-1/R} & \text{for } \lambda\xi_T \leq W_0^{-R} \\ W_0 & \text{otherwise} \end{cases},$$

where $\lambda > 0$ is chosen to satisfy the budget constraint

$$(7) \quad W_0 = E[C\xi_T].$$

This is a standard option pricing problem.[2]

Table 3 shows how the portfolio choice for preferences of the form in (5) depends on the interest rate and time to maturity. Parameter choices are motivated by the U.S. markets: $\sigma = .2$ annually and $\mu - r = .1$ annually. The interest rate r and time-to-maturity T are varied in the table. The risk-aversion parameter R is chosen to make it optimal to keep exactly half of one's wealth in equities in the limit as $T \to \infty$. Since r should be a real interest rate, a small value such as 2% or 4% is most relevant. Even with the extreme assumption that preferences are identical above W_0, the portfolio mix can be significantly different from its asymptotic value even at as long a maturity as 25 years.

[2]Consumption is equal to W_0, the payoff of a riskless bond, plus $\lambda^{-1/R}\max(\xi_T^{-1/R} - W_0\lambda^{1/R}, 0)$, the payoff of a number $\lambda^{-1/R}$ of call options with exercise price $W_0\lambda^{1/R}$ on an asset paying $\xi_T^{-1/R}$. Given λ, the value before T of receiving $\xi_T^{-1/R}$ at T follows a lognormal distribution with constant variance, so pricing is according to Black-Scholes. We compute λ by a one-dimensional search for the value satisfying the budget constraint.

Portfolio choice: infinite risk aversion below W_0							
	years to maturity						
r	1	2	5	10	25	50	100
2%	0.16	0.20	0.28	0.34	0.42	0.47	0.50
4%	0.25	0.31	0.40	0.45	0.49	0.50	0.50
6%	0.32	0.39	0.46	0.49	0.50	0.50	0.50
8%	0.37	0.43	0.48	0.50	0.50	0.50	0.50
10%	0.41	0.46	0.49	0.50	0.50	0.50	0.50

Table 3: The table gives the proportion of wealth invested initially in the risky asset. The utility function is taken to be $-\infty$ below W_0 and has constant relative risk aversion $2(\mu - r)/\sigma^2$ above W_0. The proportion of wealth in the risky asset converges to one-half as maturity increases, but slowly. The stock return has mean $r + 10\%$ per year and standard deviation 20% per year. The riskless rate r per annum and years to maturity are varied in the table.

Example 3

Our final example on convergence assumes that risk aversion is zero (rather than infinity) below W_0. The utility function has the form

$$(8) \qquad u_1(C) \equiv \begin{cases} \frac{C^{1-R}}{1-R} & \text{for } C \geq W_0 \\ \frac{{W_0}^{1-R}}{1-R} - {W_0}^{-R}(W_0 - C) & \text{for } 0 \leq C < W_0 \\ -\infty & \text{for } C < 0 \end{cases},$$

which is a power function above W_0, linear below W_0, and chosen to be continuous and differentiable at W_0. The marginal utility in this case is given by C^{-R} for $C \geq W_0$, by ${W_0}^{-R}$ for $C \in (0, W_0)$, and the range $[{W_0}^{-R}, \infty)$ at

$C = 0$. The first-order condition for an optimum implies that[3]

$$(9) \quad C = \begin{cases} (\lambda\xi_T)^{-1/R} & \text{for } \lambda\xi_T \le W_0^{-R} \\ 0 & \text{otherwise} \end{cases},$$

where $\lambda > 0$ is chosen to satisfy the budget constraint

$$(10) \quad W_0 = E[C\xi_T].$$

Again this evaluation is a simple option pricing problem[4]

Table 4 shows how the portfolio choice for preferences of the form in (8) depends on the interest rate and time to maturity. Parameter choices are motivated by the U.S. markets: $\sigma = .2$ annually and $\mu - r = .1$ annually. The interest rate r and time-to-maturity T are varied in the table. The risk-aversion parameter R is chosen to make it optimal to put exactly half of one's wealth in equities in the limit as the horizon tends to infinity. For reasonable parameter values, convergence can be very slow as before.

Failure of Turnpike Results for Consumption Withdrawal Problems

Consider the following investment problem with consumption withdrawal, written in terms of consumption (with the portfolio strategy implicitly substituted out).

[3] While C is indeterminate in $[0, W_0]$ when $\lambda\xi_T = W_0^{-R}$, the measurable selection does not affect the random variable since this occurs on a set of states of measure 0, because $\log \xi_T$ has a Gaussian distribution.

[4] Consumption can be viewed as the value of receiving at T an asset worth $(\lambda\xi_T)^{-1/R}$ at T in the even the asset is worth at least W_0 and zero otherwise. This is close relative to a call option on the asset. Since the asset's value before T is easily seen to follow a lognormal process with constant variance, Black-Scholes pricing obtains, and in fact the value of this asset is given by the first term (containing the stock price as a factor) of the Black-Scholes call option pricing formula. Finding the correct λ involves a one-dimensional search for the zero of a monotone function.

Portfolio choice: zero risk aversion below W_0							
	years to maturity						
r	1	2	5	10	25	50	100
2%	1.46	1.27	1.02	0.85	0.66	0.56	0.51
4%	1.37	1.15	0.87	0.69	0.53	0.50	0.50
6%	1.27	1.03	0.73	0.57	0.50	0.50	0.50
8%	1.17	0.91	0.62	0.52	0.50	0.50	0.50
10%	1.08	0.81	0.55	0.50	0.50	0.50	0.50

Table 4: The table gives the proportion of wealth invested initially in the risky asset. The utility function is taken to be linear below W_0 and has constant relative risk aversion $2(\mu - r)/\sigma^2$ above W_0. The proportion of wealth in the risky asset converges to one-half as maturity increases, but slowly. The stock return has mean $r + 10\%$ per year and standard deviation 20% per year. The riskless rate r per annum and years to maturity are varied in the table.

Problem 1 *Choose adapted and right-continuous* $\{c_t\}$ *to*
maximize $E[\int_0^T u(c_t)e^{-\delta t}dt]$
subject to $E[\int_0^T \xi_t c_t dt] = W_0$.

The point of this section is to show that we cannot expect to have a portfolio tunpike theorem in a consumption-withdrawal problem such as Problem 1. The reason is that while consumption in the far future may reflect growth to very large levels of consumption (depending on the relation between the impatience parameter δ and the other parameters), consumption at nearby dates reflects the shape of the utility function at relatively small consumption levels, even as the horizon increases indefinitely. This result is a reminder that when we talk about convergence of a portfolio strategy at long horizons, this should be interpreted as a long horizon until the first consumption withdrawal, not as a long horizon for the overall problem.

To make the point explicitly, take the translated power case with felicity function $u(c) = (c - K)^{1-R}/(1 - R)$, which is similar to the power function $u(c) = c^{1-R}/(1 - R)$ at large consumption levels. The cost of maintaining the lower bound to consumption $c_t \equiv K$ is given by the annuity formula $E[\int_{t=0}^{T} \xi_t K dt] = (1 - e^{-rT})K/r$, so we can solve for $c_t - K$ with the power felicity and initial wealth $W_0 - (1 - e^{-rT})K/r$. But the power function is homothetic and therefore has consumption proportional to wealth. Therefore consumption in the translated power case is given by K plus $1 - (1 - e^{-rT})K/rW_0$ times consumption in the power case, which does not converge to the power consumption as $T \to \infty$. Furthermore, the risky portfolio investment is a factor $1 - (1 - e^{-rT})K/rW_0$ times what it would be in the power case, which does not converge to the power portfolio choice either. These results depend only on fixed interest rate, existence of solutions, and some asset always having a nonzero risk premium (so the power portfolio choice is not the riskless asset).

2 Formal Model and Two Turnpike Theorems

In this section, we present two turnpike theorems in continuous time. These theorems are intended to synthesize and generalize existing results in the literature. Both results look at preferences that are similar to some benchmark that may not be power utility as in the literature. Theorem 1 puts very little restriction on security returns (beyond the underlying Brownian model), while Theorem 2 assumes less regularity on preferences but a strong assumption (i.i.d.) on security returns. With i.i.d. returns, we require less regularity on preferences, since investing to time T with nonsmooth preferences is fully equivalent to investing to an earlier time with smooth preferences.

To begin with, we specify the market, which we shall refer to as the *standard Brownian market*. Portfolio returns are defined using the standard continuous-time model of a complete securities market. There are N locally risky assets indexed by $n \in \{1, 2, ..., N\}$ and a single locally riskless asset. The underlying uncertainty is modeled by the complete filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, P)$ generated by an N-dimensional Wiener process $\{Z_t | t \in [0, \infty)\}$ with independent components, and all processes are adapted to $(\mathcal{F}_t)_{t \geq 0}$. Conditional expectation with respect to $\mathcal{F}_t$ will be denoted by

E_t. The riskless asset bears an interest rate following a process r and local returns to the risky assets are given by

$$(11) \quad \mu_t dt + \sigma_t dZ_t,$$

where the N-vector process μ gives the mean returns and the non-singular $N \times N$ matrix process σ relates the random part of stock returns to the underlying sources of noise. We denote by $\Sigma \equiv \sigma\sigma'$ the covariance matrix of returns. [5] We define the *discount* process by

$$(12) \quad \beta_t \equiv \exp\left(-\int_{\tau=0}^{t} r_\tau d\tau\right),$$

and the *risk-neutral change-of-measure* process by

$$(13) \quad \rho_t \equiv \exp\left(-\int_{\tau=0}^{t} \gamma_\tau' dZ_\tau - \frac{1}{2}\int_{\tau=0}^{t} |\gamma_\tau|^2 d\tau\right),$$

where $\gamma_t \equiv \sigma_t^{-1}(\mu_t - r_t\mathbf{1})$, and $\mathbf{1}$ is a vector of ones. By its definition, ρ is a local martingale; we shall assume

$$(14) \quad \rho \quad \text{is a martingale,}$$

so that we may consistently define the risk-neutral probability measure Q by

$$(15) \quad E_t^Q[x] \equiv E[\rho_t x]$$

[5] Nonsingularity of σ and an equal number of assets and sources of noise is a convenience. What is actually needed to avoid arbitrage is that the vector $\mu - r\mathbf{1}$ of excess returns must be in the span of the columns of σ to ensure that priced risk has positive variance. For Theorem 1, we also require that σ should have full column rank for completeness. (For Theorem 2, essential completeness, over the states distinguished by security returns, is always true even if σ does not have full column rank.) The theorems and proofs are otherwise the same except using appropriate left-inverses or generalized inverses.

for any bounded $\mathcal{F}_t$-measurable random variable x. As a last piece of notation, we shall define the *state-price density* process

(16) $\xi_t \equiv \rho_t \beta_t.$

We make the regularity assumption that for all t, ξ_t has all moments, positive and negative:

(17) $(\forall \eta \in \Re, t < \infty)\ E[\xi_t^\eta] < \infty.$

This is a relatively modest assumption, which would follow if r and $\sigma^{-1}(\mu - r\mathbf{1})$ were assumed to be bounded processes (which would also suffice to make ρ a martingale.) This completes the definition of the standard Brownian market.

Within this framework, the wealth process w_t of an agent who at time t holds the vector θ_t of dollar investments in the locally risky assets satisfies

(18) $w_t \equiv W_0 + \int_{\tau=0}^{t} (r_\tau w_\tau d\tau + \theta'_\tau \{(\mu_\tau - r_\tau \mathbf{1}) d\tau + \sigma_\tau dZ_\tau\}),$

where W_0 is the agent's initial wealth, and we require the nonnegative wealth and consumption constraints

(19) $(\forall t)\, w_t \geq 0$

and

(20) $C \leq w_T,$

which rule out borrowing without repayment, doubling strategies and related arbitrages. In terms of the *discounted wealth process*

(21) $\tilde{w}_t \equiv \beta_t w_t,$

the budget equation (18) takes the simple form

$$(22)\quad \tilde{w}_t = W_0 + \int_{\tau=0}^{t} \tilde{\theta}'_\tau\{(\mu_\tau - r_\tau \mathbf{1})d\tau + \sigma_\tau dZ_\tau\},$$

where $\tilde{\theta} \equiv \beta\theta$. Applying Itô's lemma to $\rho_t w_t$ shows that the discounted wealth process is a Q-local martingale.

A typical agent solves the following problem.

Problem 2 *Choose C and adapted $\{\theta_t\}$ to maximize $Eu(C)$ subject to (18), (19) and (20).*

The horizon T is fixed, but is thought of as extremely large, and where the von Neumann-Morgenstern (vN-M) utility function u of the agent is *convenient* in the sense now to be defined. A utility function $u : (\underline{C}, \infty) \to \Re$ is said to be *convenient* if it is strictly increasing and strictly concave, and has continuous first derivative, with $u(\underline{C})$ equal to the right limit at $\underline{C}$ if the limit exists. To manage the boundary at $\underline{C}$, whether or not the derivative is finite there, we will consider the derivative correspondence (or support correspondence)[6] defined by

$$(23)\quad u'(C) \equiv \{m \in \Re | (\forall D > \underline{C}) u(D) \le u(C) + m(D - C)\},$$

where we identify the set containing a single element with the element to allow us the usual notation at points of differentiability (which are all $C > \underline{C}$ given our assumptions).

Here then is the main result of the paper.

[6] Using the derivative correspondence can handle interior points of nondifferentiability as well as boundary points, although to simplify our theorems we restrict ourselves to utility functions that are differentiable on the interiors of their domains.

Theorem 1 *Consider two agents 0 and 1 with convenient utilities u_0 and u_1 respectively, with common initial wealth W_0, each solving Problem 2 in the standard Brownian market. Assume that the market grows indefinitely:* [7]

$$(24)\quad \lim_{T\to\infty} E[\xi_T] = 0,$$

and that the horizon T is large enough that the initial wealth will satisfy the subsistence requirements of both agents at that time, in the sense that for $i = 1, 0$

$$(25)\quad W_0 > E[\xi_T]\underline{C}_i.$$

Assume that the two utilities are similar at infinity in the precise sense [8] *that*

$$(26)\quad \lim_{C\to\infty} \frac{u_1'(C)}{u_0'(C)} = 1,$$

and moreover that the utilities have the uniform continuity property that for all sequences a_n, $b_n \to \infty$,

$$(27)\quad \frac{b_n}{a_n} \to 1 \text{ iff } \frac{u_i'(b_n)}{u_i'(a_n)} \to 1.$$

If $w_{it;T}$ denotes the optimal wealth process of agent i with horizon T, and if $\theta_{i\tau;T}$ deonotes the corresponding portfolio process, then for each $t > 0$

$$(28)\quad \lim_{T\to\infty} E^Q|\tilde{w}_{0t;T} - \tilde{w}_{1t;T}| = 0,$$

[7] This condition is that the riskless discount factor (the value today of one dollar at maturity) goes to zero as maturity increases. This is certainly valid if the interest rate is constant and positive, and it would appear to be a feature of any reasonable term structure model.

[8] This is the same as requiring that for each representation, the ratio of marginal utilities tends to a constant; in the proofs we will take the constant to be 1 so that (26) holds.

and

$$(29)\quad \operatorname*{plim}_{T\to\infty} \int_{\tau=0}^{t} (\theta_{1\tau;T} - \theta_{0\tau;T})'\Sigma_\tau(\theta_{1\tau;T} - \theta_{0\tau;T})\, d\tau = 0.$$

(We think of the plim *as being taken in actual probabilities P, but of course this is equivalent to taking the* plim *in the equivalent probability measure Q.)*

Moreover,

$$(30)\quad \operatorname*{plim}_{T\to\infty} \sup_{s\le t} |w_{1s;T} - w_{0s;T}| = 0.$$

The formal proof of this result is in the Appendix, and we provide a sketch of the proof in the text, but first we comment briefly on the conditions on preferences and specifically the uniform continuity condition, which is a very weak condition, especially compared with the conditions in the literature. First note that the Inada condition

$$(31)\quad \lim_{C\to\infty} u_i'(C) = 0$$

follows directly from the uniform continuity property and strict concavity of u_i. Then, to see the sense in which (27) is a uniform continuity property, for any convenient u satisfying the Inada condition (31) with corresponding $\underline{C}$, choose any $a > log(\max(\underline{C}, 1))$ and define $f : [a, \infty) \to \Re$ by $f(x) = \log u'(e^x)$. This function f is the function we are plotting if we plot $u'(\cdot)$ with logarithmic scales on the axes. Given (31) and that u is convenient, this is a continuous and strictly decreasing function and hence has a continuous inverse with domain $f[a, \infty) = (-\infty, f(a)]$. Assumption (27) is equivalent to uniform continuity of f and f^{-1} on their domains, given the assumptions (positivity, continuity, strict monotonicity, and the Inada condition) already made about u_i'.

The CRRA case of the existing literature is the special case of linear f and f^{-1}, but (27) also holds for a significantly larger class of functions, including, for example, all twice continuously differentiable functions whose relative risk aversion is bounded above and bounded below away from zero, as well as all u for which u' varies regularly at infinity with exponent $-R < 0$. In effect, regular variation would say that the utility function looks similar to a power function at large consumption levels; for our condition (27) it suffices for the function to look similar to different power functions in a bounded set of powers along different sequences of large wealth levels. While we have stated that this condition must hold symmetrically for both utility functions, it suffices to assume it for one utility function, since it must then follow for the other given (26); since it is satisfied by CRRA preferences, a much stronger form of this assumption has been assumed in all the previous literature. For the reader wishing to know more about the connection between this assumption and other forms of regularity, we offer the following Lemma which is proven in the Appendix.[9]

Lemma 1 *Assume a convenient utility function satisfying the Inada condition (31). Then the implications*

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii)

hold among the statements (i)–(vii) defined below. As above, $f(x) \equiv \log(u'(\exp(x)))$ *and a is any number larger than* $\log(\max(\underline{C}, 1))$.

(i) Relative risk aversion converges as wealth increases: u is twice continuously differentiable with $(\forall C)u''(C) > 0$ *and* $\lim_{C\uparrow\infty} -Cu''(C)/u'(C) = R^* > 0$.

[9] On a minor technical point, Huberman and Ross assume the marginal utility function is regularly varying with index $-R$, $0 < R < 1$. They state that this is equivalent to relative risk aversion converging to R as wealth tends to infinity. However, convergence of relative risk aversion is a stronger condition. Consider a utility function defined for $x > \underline{C} \geq 0$ as an integral of the marginal utility function $u'(x) = x^{-R}\exp(-\gamma \sin x/x)$ for some constant γ. For γ sufficiently close to zero, one can show that $u'' < 0$, so the utility function is a strictly monotone, concave function. This marginal utility function is regularly varying at infinity with coefficient R. This means that $\lim_{x\to\infty} u'(ax)/u'(x) = a^{-R}$ for all $a > 0$. However the coefficient of relative risk aversion is $R + \gamma\cos x - \gamma \sin x/x$, which does not converge to R as $x \to \infty$.

It is important to note that the counterexample does not affect Huberman and Ross's main result, which assumed the weaker condition of regular variation.

(ii) Relative risk aversion bounded above and below away from zero: u is twice continuously differentiable and $(\exists \underline{R}, \overline{R})(\forall C \in [\exp(a), \infty))(0 < \underline{R} < -Cu''(C)/u'(C) < \overline{R})$.

(iii) Lipschitz condition on f and f^{-1}*:* $(\exists \underline{K}, \overline{K} > 0)(\forall x, y \in [\log(a), \infty), y > x)(\underline{K}(y-x) \leq f(x) - f(y) \leq \overline{K}(y-x))$. *(Note that this expression combines the Lipschitz conditions for f and* f^{-1} *given that we know* $f' < 0$*.)*

(iv) Declines in marginal utility are bounded above and below by power functions: $(\exists k, k')(\forall C \in [\exp(a), \infty), \forall C' > C)(1 > (C/C')^k > u'(C')/u'(C) > (C/C')^{k'})$.

(v) Uniform continuity of f and f^{-1}*:* $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x, y \in [a, \infty))((|x - y| < \delta) \Rightarrow (|f(x) - f(y)| < \varepsilon))$ *and the analogous condition for* f^{-1}.

(vi) Condition (27) for all sequences $\{a_n\}, \{b_n\}$ *taking values in* $[a, \infty)$.

(vii) Condition (27) for all sequences $\{a_n\}, \{b_n\} \to \infty$ *taking values in* $[a, \infty)$.

PROOF The formal proof is in the Appendix. One of the critical observations in the proof is that when u is twice differentiable, $f'(x) = e^x u''(e^x)/u'(e^x)$, which provides the link between f and the relative risk aversion. ∎

Now we sketch the proof of Theorem 1; the formal proof is in the Appendix. The proof is in six steps. The first and third steps are by now standard (see, for example, Karatzas (1989)), but we include them for completeness.

The first step shows that the budget constraint and nonnegative wealth constraint can be collapsed to a "static" budget constraint, $E[\xi_T w_T] \leq W_0$.

The second step shows that the similarity and uniform continuity of marginal utility functions imply corresponding properties of the inverse marginal utility functions.

The third step constructs and characterizes the unique optimum for each agent; as is well known, agent i's optimal wealth for horizon T can be expressed as $w_{iT;T} = I_i(\lambda_{iT}\xi_T)$ for some $\lambda_{iT} > 0$, where I_i is the inverse to u_i'.

The fourth step establishes convergence of the Lagrange multipliers that characterize the optima; $\lim_{T\to\infty} \frac{\lambda_{0T}}{\lambda_{1T}} = 1$.

The fifth step shows the two wealth processes converge (28), and the last step deduces the remaining statements (29) and (30) from (28) using the Burkholder-Davis-Gundy inequalities.

As announced in the Introduction, we also state here the second main result of this paper, which is the continuous-time analogue of the result of Huberman and Ross (1983) albeit with less smoothness assumed for the preferences. This result assumes i.i.d. returns and a benchmark portfolio that exhibits CRRA (as did Huberman and Ross), but the sense of similarity (33) is much weaker the sense (26) in Theorem 1. One particular innovation in the proof is the smoothing of preferences by randomization that comes from looking at a slightly different time horizon: the portfolio choice for preferences for which $u''(\cdot)$ does not exist is the same as the portfolio choice of a very smooth utility function at a slightly shorter horizon. This proof technique allows us to construct a proof for the very smooth case (based on Fourier inversion and exact formulas for the wealth and portfolio processes) and then use the smoothing to extend the proof to the general case when preferences are less smooth.

Theorem 2 *Consider two agents in a standard Brownian market in which the processes r, σ and μ are all deterministic, and in which the market growth condition (24) holds. Suppose that agent 0 has marginal utility*

(32) $u_0'(C) = C^{-R} \qquad C > 0,$

where $R > 0$ (corresponding to power or log utility), and that agent 1 has a convenient utility function whose marginal utility varies regularly at infinity *with exponent $-R$, which is to say*

$$(33) \quad (\forall a > 0) \lim_{C \uparrow \infty} \frac{u_1'(aC)}{u_1'(C)} = a^{-R}.$$

The agents start with the same initial wealth W_0 and solve Problem 2. (i) Then for large horizons the optimal wealth processes are close in the sense that for all $t > 0$

$$(34) \quad \lim_{T \to \infty} E^Q |\tilde{w}_{0t;T} - \tilde{w}_{1t;T}|^2 = 0,$$

from which we deduce that we even have

$$(35) \quad \lim_{T \to \infty} E^Q [\sup_{s \in [0,t]} (w_{1s;T} - w_{0s;T})^2] = 0,$$

and convergence of portfolios

$$(36) \quad \lim_{T \to \infty} E^Q [\operatorname{ess\,sup}_{s \in [0,t]} |\theta_{1s;T} - \theta_{0s;T}|^2] = 0.$$

(ii) The portfolio strategy and wealth process for agent 0 do not depend on the horizon T and can be written as

$$(37) \quad \theta_{0t} = w_{0t} R^{-1} \Sigma_t^{-1} (\mu_t - r_t \mathbf{1})$$

and

$$(38) \quad w_{0t} = W_0 \xi_t^{-1/R} \exp\left(-(R^{-1} - 1) \int_{s=0}^{t} (r_s + \kappa_s / 2R) ds \right)$$

where

$$(39)\quad \kappa_s \equiv (\mu_s - r_s\mathbf{1})'\Sigma_s^{-1}(\mu_s - r_s\mathbf{1}).$$

Agent 1's portfolio proportions converge to agent 0's proportions in the sense that for each $t > 0$

$$(40)\quad \operatorname*{ess\,sup}_{s\in[0,t]} \left| \frac{\theta_{1s;T}}{w_{1s;T}} - R^{-1}\Sigma_s^{-1}(\mu_s - r_s\mathbf{1}) \right| \xrightarrow{T\uparrow\infty} 0,$$

in probability.

The proof of this result is also in the Appendix, but we give here some comments on the conditions, and an outline of the strategy of the proof. The assumption of a *deterministic* standard Brownian market is the analogue of the assumption of independent returns used by Huberman and Ross in their discrete time result; the returns in the deterministic standard Brownian market are indeed independent over disjoint time intervals. We cannot apply Theorem 1 because (for example) the utility u_1 for which $u_1'(x) = x^{-R}/\log(2+x)$ satisfies (33) but the comparison condition (26) needed for Theorem 1 fails. It is not surprising that the conclusions of Theorem 2 are stronger than those of Theorem 1, in view of the stronger assumptions; however, we conjecture that the main conclusion (28) of Theorem 1 may remain true even if the utilities satisfy only the less stringent conditions (32) and (33) of Theorem 2 rather than (26).

The essential part of the proof is to notice that in the deterministic standard Brownian market, the expression $w_{iT;T} = I_i(\lambda_{iT}\xi_T)$ and the fact that the discounted optimal wealth process is a Q-martingale allow us to write the optimal wealth process as $w_{s;T} = h(\xi_s, s, T)$ where $h(x, s, T) \equiv E[\xi_{sT} I(x\lambda_T\xi_{sT})]$ and $\xi_{sT} \equiv \xi_T/\xi_s$ is the state-price density for purchase at s of a claim at T. By carrying out the Itô expansion of the optimal wealth process, we can identify the optimal portfolio, which we deduce must be

$$(41)\quad \theta_{s;T} = -\xi_s h_x(\xi_s, s, T)\Sigma_s^{-1}(\mu_s - r_s\mathbf{1}).$$

Now if we formally differentiate the expression for h with respect to x, we find that

$$(42)\quad \begin{aligned} h_x(x,s,T) &= E[\lambda_T \xi_{sT}^2 I'(x\lambda_T\xi_{sT})] \\ &= -x^{-1}E[\xi_{sT}\frac{I(x\lambda_T\xi_{sT})}{R(I(x\lambda_T\xi_{sT}))}], \end{aligned}$$

where $R(\cdot)$ is the familiar risk aversion function defined to be $R(x) \equiv -xu''(x)/u'(x)$. The form of agent 0's optimal portfolio now follows (since R is constant); the asymptotic similarity of the policies for the two agents requires analysis of h_x for agent 1, showing that the main part of the expectation is due to sample paths for which $R(I(x\lambda_T\xi_{sT}))$ is very close to R.

There is a technical point in the proof, namely that the formal differentiation of h cannot lead to any expression of the form we have given *if I is not differentiable*; and we have made no assumption of differentiability of I_1. The way round this point is to introduce a smoothed version of the utility of agent 1, smoothed in a cunning way so that the optimal behaviour of the original agent 1 and the smoothed agent 1 agree on $[0,t]$. The auxiliary results needed to deal with this are given separately as Lemma 2 and Proposition 1.

3 Conclusion

Portfolio turnpike theorems are interesting conceptually because they describe the limiting behavior of portfolio strategies as the investment horizon increases. Unfortunately, their practical importance is limited by the slow rate of convergence.

4 Appendix

PROOF OF LEMMA 1 : Recall that a utility function $u : (\underline{C}, \infty) \to \Re$ is said to be *convenient* if it is strictly increasing and strictly concave, and has continuous first derivative, with $u(\underline{C})$ equal to the right limit at $\underline{C}$ if the limit exists.

(i) $\Rightarrow$ (ii): Under the assumptions, $R(C) \equiv -Cu''(C)/u'(C)$ is continuous on $[a, \infty)$ and converges at ∞ to $R^* > 0$, which implies that $R(\cdot)$ is bounded on the whole interval. The smallest value is either $R^* > 0$ achieved in the limit as C increases, or it is achieved at some finite C^*. Since $u''(C^*) < 0$ and $u'(C^*) > 0$, the minimum is positive.

(ii) $\Rightarrow$ (iii): Since $f'(x) = \exp(x)u''(\exp(x))/u'(\exp(x)) = -R(\exp(x))$, $0 > -\underline{R} < f'(x) < -\overline{R}$, and f Lipschitz follows from integrating this expression. Similarly, the derivative of $f^{-1}(x)$ is $1/(f' \circ f^{-1})(x)$, which is bounded between $-1/\overline{R}$ and $-1/\underline{R}$, from which it follows that f^{-1} is Lipschitz.

(iii) $\Leftrightarrow$ (iv): This follows immediately from substitution of the definition of f.

(iii) $\Rightarrow$ (v): Simply take $\delta = \epsilon/\overline{K}$ to prove uniform continuity of f, or $\delta = \varepsilon\underline{K}$ to prove uniform continuity of f^{-1}.

(v) $\Leftrightarrow$ (vi): Actually, the "only if" part of (27) is equivalent to uniform continuity of f, and the "if" part of (27) is equivalent to uniform continuity of f^{-1}. We prove the equivalence for the "only if" part of (27) and uniform continuity of f; the proof of the other part is identical. Letting $x_n \equiv \log(a_n)$ and $y_n \equiv \log(b_n)$ and using the definition of f, the "only if" part of (27) is equivalent to $|y_n - x_n| \to 0 \Rightarrow |f(y_n) - f(x_n)| \to 0$. But this is just uniform continuity of f by definition of the limits.

(vi) $\Leftrightarrow$ (vii): This equivalence follows because the continuity of f and f^{-1} implies uniform continuity on compact sets. Therefore, any failure of one of the limits must happen on an unbounded pair of sequences, which can be taken without loss of generality (by taking an increasing subsequence) to tend to ∞. Conversely, if we have convergence on all unbounded pairs of sequences tending to ∞, uniform continuity on compact sets implies convergence for all sequences.

∎

PROOF OF THEOREM 1

The proof contains the 6 steps described in the text.

Step 1: Feasible consumption We will omit the subscripts i for the agent and T for the horizon in this part. Given the horizon T, the set of random terminal consumptions C consistent with (18)–(19) is the set of nonnegative random variables C satisfying

$$(43)\quad E[\xi_T C] \leq W_0.$$

The necessity of (43) follows from applying Itô's lemma to $\xi_t w_t$ (as defined in (16) and (18)) and observing that it is a local martingale and by non-negativity therefore a supermartingale. Consequently, by (20), $E[\xi_T C] \leq E[\xi_T w_T] \leq E[\xi_0 w_0] = W_0$. Conversely, if non-negative C satisfies (43), set $w_T = C + (W_0 - E[\xi_T C])/E[\xi_T]$. Then $E[\xi_T w_T] = W_0$. Let $W_0 + \int_{\tau=0}^{t} \phi'_\tau dZ_\tau$ be the predictable representation of the martingale $M_t \equiv E_t[\xi_T w_T]$, where E_t indicates expectations based on information (Z_s for $0 \leq s \leq t$) known at t. Set $w_t = \xi_t^{-1} M_t$. Then (20) and (19) follow from (43) and positivity of ξ, and w and $\theta_t \equiv \xi_t^{-1}(\sigma')^{-1}\phi + w_t \Sigma_t^{-1}(\mu_t - r_t \mathbf{1})$ satisfy (18).

Step 2: Inverse marginal utility functions Agent i's inverse marginal utility function

$$(44)\quad I_i(x) \equiv \begin{cases} (u_i')^{-1}(x) & \text{for } x < \lim_{C \downarrow \underline{C}_i} u_i'(C) \\ \underline{C}_i & \text{otherwise} \end{cases}$$

will play an important role in the analysis. Note that u_i' may be a correspondence but not a function, since if $\lim_{C \downarrow \underline{C}_i} u_i'(C) < \infty$, $u_i'(\underline{C}_i) =$

$[\lim_{C\downarrow \underline{C}_i} u_i'(C), \infty)$. However, by positivity, continuity, and monotonicity of u_i' and the Inada condition (31), $I_i(x)$ is a well-defined and continuous function for all positive x. Condition (27) on the marginal utilities implies an analogous property for the inverse marginal utilities: for all sequences x_n, $y_n \downarrow 0$,

$$(45)\quad \frac{I_i(y_n)}{I_i(x_n)} \to 1 \text{ iff } \frac{y_n}{x_n} \to 1.$$

Given monotonicity of u_i' and the Inada condition (31), this follows immediately from (27) if we set $b_n \equiv I_i(y_n)$ and $a_n \equiv I_i(x_n)$. And, given (45), similarity (26) implies a similar condition on inverse marginal utility functions:

$$(46)\quad \lim_{x\downarrow 0} \frac{I_1(x)}{I_0(x)} = 1.$$

To see this, note that

$$(47)\quad \begin{aligned} \frac{I_1(x)}{I_0(x)} &= \frac{I_0(u_0'(I_1(x)))}{I_0(x)} \\ &= \frac{I_0(\frac{u_0'(I_1(x))}{u_1'(I_1(x))}x)}{I_0(x)}. \end{aligned}$$

As $x \downarrow 0$, $u_0'(I_1(x))/u_1'(I_1(x))$ converges to 1 by (26), so the expression in (47) converges to 1 by (45).

An additional implication of (45) is that $I(x)$ grows no faster than a power of x as $x \downarrow 0$. Specifically, (45) implies that there exists $\gamma < 1$ and $\varepsilon > 0$ such that

$$(\forall x \in (0,\varepsilon), y \in (\gamma x, x))\ \frac{I(y)}{I(x)} \leq e.$$

Otherwise, there would exist $\gamma_n \uparrow 1$, $x_n \downarrow 0$ and $\gamma_n \leq y_n/x_n \leq 1$ such that $I(y_n)/I(x_n) > e > 1$, which would contradict the "if" part of (45). This implies that for $x < \varepsilon$,

$$I(x) = I(\varepsilon)\frac{I(\gamma\varepsilon)}{I(\varepsilon)}\frac{I(\gamma^2\varepsilon)}{I(\gamma\varepsilon)}\cdots\frac{I(\gamma^N\varepsilon)}{I(\gamma^{N-1}\varepsilon)}\frac{I(x)}{I(\gamma^N\varepsilon)} \leq I(\varepsilon)e^{N+1},$$

where N is defined by

$$\varepsilon\gamma^{N+1} \leq x \leq \varepsilon\gamma^N.$$

This, in conjunction with monotonicity, yields

(48) $(\forall x > 0)\ I(x) \leq A + Bx^{1/\log\gamma}$

for some constants A and B, i.e. $I(x)$ is bounded by a constant plus a power times a constant.

Step 3: Existence and characterization of unique optimal demand Again, we omit subscripts indicating the agent and the horizon. The optimal consumption for an agent maximizes $Eu(C)$ among random variables bounded below by $\underline{C}$ subject to the constraint (43). The first-order necessary conditions for this optimization are

(49) $(\exists\lambda \geq 0)\ C = I(\lambda\xi_T)$

together with the constraint (43) as an equality. To see that this is sufficient whether or not $\lim_{C\downarrow\underline{C}} u'(C)$ is finite, note that $\lambda\xi_T$ is always a member of the derivative correspondence $u'(I(\lambda\xi_T))$ and therefore for any other random consumption D satisfying the budget constraint $E[D\xi_T] \leq W_0$, $Eu(D) \leq E[u(I(\lambda\xi_T)) + \lambda\xi_T(D - C)] \leq E[u(I(\lambda\xi_T))]$, where the last inequality follows from the budget constraints. To verify that $E[u(I(\lambda\xi_T))]$ is finite, set $D = W_0/E[\xi_T]$ to compute a lower bound, and substitute in the support (23) at $C = \underline{C} + 1$ to apply (48) and (17) to compute an upper bound. (As with other variables, λ varies with the agent and horizon, but we are suppressing this dependence.) Therefore, the problem reduces to one of finding λ such that [10]

$$(50)\quad E[\xi_T I(\lambda\xi_T)] = W_0.$$

Since $I(x)$ is bounded by a constant plus a power times a constant and ξ possesses all moments, $E[\xi_T I(\lambda\xi_T)] < \infty$ for all λ, and by Lebesgue's monotone convergence theorem, $E[\xi_T I(\lambda\xi_T)]$ is a continuous function of λ.

The function $\lambda \mapsto f(\lambda) \equiv E[\xi_T I(\lambda\xi_T)]$ maps $(0,\infty)$ into $(\underline{C}E[\xi_T],\infty)$, is unbounded (because of the Inada condition), and is strictly decreasing in the open interval where $f > \underline{C}E[\xi_T]$; therefore, the assumption that wealth is greater than the present value of subsistence consumption (25) implies that there exists a unique λ satisfying (50).

To summarize, there exists a unique optimal consumption given by (49) where λ is the unique solution to (50). This optimal consumption is generated by the portfolio policy described in the derivation of (43). The uniqueness of the portfolio policy follows from uniqueness of the predictable representation and nonsingularity of σ.

Step 4: Convergence of Lagrange multipliers Let λ_{iT} denote the Lagrange multiplier described in the previous step for agent i with horizon T. We will show that

$$(51)\quad \lim_{T\to\infty} \frac{\lambda_{0T}}{\lambda_{1T}} = 1.$$

By symmetry, it suffices to show that $\liminf_{T\to\infty} \lambda_{0T}/\lambda_{1T} \geq 1$.

[10] This omits one degenerate case, corresponding intuitively to $\lambda = \infty$, in which $\underline{C}E[\xi_T] = W_0$ and $u(\underline{C})$ is well-defined. In that case, the optimum is the only feasible strategy, for which consumption is $\underline{C}$. For the turnpike result, $E[\xi_T]$ tends to 0 as maturity increases, and we have that the degenerate case never arises for sufficiently large T.

Suppose to the contrary that $\liminf_{T\to\infty} \lambda_{0T}/\lambda_{1T} < 1$. Then there exists $\delta < 1$ and an unbounded set $\mathcal{T}$ of terminal times such that $(\forall T \in \mathcal{T})\lambda_{0T}/\lambda_{1T} \leq \delta$. For $T \in \mathcal{T}$,

$$
\begin{aligned}
(52)\quad W_0 &= E[\xi_T I_0(\lambda_{0T}\xi_T)] \\
&\geq E[\xi_T I_0(\delta\lambda_{1T}\xi_T)] \\
&\geq E[\xi_T I_0(\delta\lambda_{1T}\xi_T) : \lambda_{1T}\xi_T < \varepsilon],
\end{aligned}
$$

for any $\varepsilon \geq 0$, where the notation $E[z : A]$ denotes the integral of the random variable z over the event A.

We claim that (45) implies the existence of $\kappa > 1$ and $\varepsilon > 0$ such that

$$(\forall x \in (0, \varepsilon))\ \frac{I_0(\delta x)}{I_0(x)} \geq \kappa.$$

To see this, note that otherwise there would exist $x_n \downarrow 0$ and $\kappa_n \downarrow 1$ such that

$$1 \leq \frac{I_0(\delta x_n)}{I_0(x_n)} \leq \kappa_n \downarrow 1,$$

which would contradict the "only if" part of (45).

Condition (46) guarantees that by taking ε sufficiently small we can ensure that $I_0(x)/I_1(x) \geq 1/\sqrt{\kappa}$, so we have

$$(\forall x \in (0, \varepsilon))\ \frac{I_0(\delta x)}{I_1(x)} \geq \sqrt{\kappa} > 1.$$

Applying this to (52) gives

$$
\begin{aligned}
W_0 &\geq \sqrt{\kappa} E[\xi_T I_1(\lambda_{1T}\xi_T) : \lambda_{1T}\xi_T < \varepsilon] \\
&= \sqrt{\kappa} W_0 - \sqrt{\kappa} E[\xi_T I_1(\lambda_{1T}\xi_T) : \lambda_{1T}\xi_T \geq \varepsilon] \\
&\geq \sqrt{\kappa} W_0 - \sqrt{\kappa} I_1(\varepsilon) E[\xi_T : \lambda_{1T}\xi_T \geq \varepsilon] \\
&\geq \sqrt{\kappa} W_0 - \sqrt{\kappa} I_1(\varepsilon) E[\xi_T] \\
&\to \sqrt{\kappa} W_0,
\end{aligned}
$$

where we have also used, successively, (50), the monotonicity of I_1, the nonnegativity of ξ_T, and (24). The contradiction $W_0 \geq \sqrt{\kappa} W_0$ shows that $\liminf_{T\to\infty} \lambda_{0T}/\lambda_{1T} \geq 1$ and by symmetry that $\lim_{T\to\infty} \lambda_{0T}/\lambda_{1T} = 1$.

Step 5: Convergence of wealth processes In this step, we will establish condition (28). Recall from Steps 1 and 3 that the optimal wealth process of agent i is

$$(53) \quad w_{it;T} \equiv \xi_t^{-1} E_t[\xi_T I_i(\lambda_{iT}\xi_T)],$$

so (28) will follow (by the conditional version of Jensen's inequality) from

$$(54) \quad \lim_{T\to\infty} E[\xi_T |I_0(\lambda_{0T}\xi_T) - I_1(\lambda_{1T}\xi_T)|] = 0.$$

Consider any $\gamma > 0$. By (45), there exists $\delta < 1$ and $\varepsilon > 0$ such that

$$(\forall x \in (0,\varepsilon)) \ \left|\frac{I_0(\delta^{-1}x)}{I_0(x)} - 1\right| \leq \gamma \text{ and } \left|\frac{I_0(\delta x)}{I_0(x)} - 1\right| \leq \gamma.$$

Otherwise, there would exist $\delta_n \uparrow 1$ and $x_n \downarrow 0$ such that either

$$\left|\frac{I_0(\delta_n^{-1}x_n)}{I_0(x_n)} - 1\right| > \gamma \text{ or } \left|\frac{I_0(\delta x_n)}{I_0(x_n)} - 1\right| > \gamma,$$

and either case would violate the "if" part of (45). By (46), we can take ε sufficiently small that

$$(\forall x \in (0,\varepsilon)) \ \left|\frac{I_1(x)}{I_0(x)} - 1\right| \leq \gamma,$$

so

$$(55) \quad (\forall x \in (0,\varepsilon)) \ \frac{I_0(\delta^{-1}x)}{I_1(x)} \geq (1-\gamma)^2 \text{ and } \frac{I_0(\delta x)}{I_1(x)} \leq (1+\gamma)^2.$$

By Step 4, there exists T_0 such that

$$(\forall T \geq T_0)\ \delta \leq \frac{\lambda_{0T}}{\lambda_{1T}} \leq \delta^{-1}.$$

When $\lambda_{1T}\xi_T \geq \varepsilon$, we have

$$0 \leq I_1(\lambda_{1T}\xi_T) \leq I_1(\varepsilon),$$

for $T \geq T_0$,

$$0 \leq I_0(\lambda_{0T}\xi_T) \leq I_0(\delta\varepsilon).$$

Hence,

$$E\left[\xi_T|I_1(\lambda_{1T}\xi_T) - I_0(\lambda_{0T}\xi_T)| : \lambda_{1T}\xi_T \geq \varepsilon\right] \leq (I_1(\varepsilon) + I_0(\delta\varepsilon))E\xi_T \to 0,$$

as $T \to \infty$, by (24).

When $x \equiv \lambda_{1T}\xi_T < \varepsilon$ and $T \geq T_0$, we have from (55) that

$$\frac{I_0(\lambda_{0T}\xi_T)}{I_1(x)} \geq \frac{I_0(\delta^{-1}x)}{I_1(x)} \geq (1-\gamma)^2,$$

and

$$\frac{I_0(\lambda_{0T}\xi_T)}{I_1(x)} \leq \frac{I_0(\delta x)}{I_1(x)} \leq (1+\gamma)^2.$$

Therefore,

$$\left|\frac{I_0(\lambda_{0T}\xi_T)}{I_1(\lambda_{1T}\xi_T)} - 1\right| \leq (1+\gamma)^2 - 1.$$

It follows that

$$\begin{aligned}
&E\left[\xi_T|I_1(\lambda_{1T}\xi_T) - I_0(\lambda_{0T}\xi_T)| : \lambda_{1T}\xi_T < \varepsilon\right] \\
&\quad\leq \left(\gamma^2 + 2\gamma\right) E\left[\xi_T I_1(\lambda_{1T}\xi_T) : \lambda_{1T}\xi_T < \varepsilon\right] \\
&\quad\leq \left(\gamma^2 + 2\gamma\right) W_0,
\end{aligned}$$

using (50) for the last inequality. Since γ can be taken arbitrarily small, this establishes (54).

Step 6: Convergence of portfolio processes Fix t. From Steps 1 and 3 and the definitions, the process $\beta_t w_{it;T}$, $0 \le t \le T$, is a Q-martingale for each i and T and is given by

$$(56) \quad \beta_t w_{it;T} = W_0 + \int_{\tau=0}^{t} \beta_\tau \theta'_{i\tau;T} \sigma_\tau dZ^Q_\tau,$$

where $dZ^Q_s \equiv dZ_s + {\sigma_s}^{-1}(\mu_s - r_s \mathbf{1})ds$ is a Q-Wiener process. Hence,

$$\Delta_{t;T} \equiv \beta_t (w_{1t;T} - w_{0t;T})$$

is a Q-martingale, and its quadratic variation from 0 to t is

$$(57) \quad [\Delta_T]_t = \int_{\tau=0}^{t} \beta^2_\tau (\theta_{1\tau;T} - \theta_{0\tau;T})' \Sigma_\tau (\theta_{1\tau;T} - \theta_{0\tau;T})\, d\tau$$

It suffices to to show that this quadratic variation converges in probability to 0 as $T \to \infty$, because

$$(58) \quad \int_{\tau=0}^{t} (\theta_{1\tau;T} - \theta_{0\tau;T})' \Sigma_\tau (\theta_{1\tau;T} - \theta_{0\tau;T})\, d\tau \le [\Delta_T]_t \sup_{\tau \in [0,t]} \beta_\tau^{-2},$$

where the supremum is finite since β is a continuous and positive process.

Let $\Delta^*_{t;T}$ denote $\sup_{\tau \le t} |\Delta_{\tau;T}|$, and consider any $p \in (0,1)$. The Burkholder-Davis-Gundy inequalities (see, e.g., Rogers and Williams [1987], IV.42) yield, for some absolute constant c_p,

$$E\left(([\Delta_T]_t)^{p/2}\right) \le c_p E\left[(\Delta^*_{t;T})^p\right].$$

Convergence of $[\Delta_T]_t$ to 0 in $L^{p/2}(\Omega, \mathcal{F}_t, Q)$ will imply the desired convergence in probability (in Q and therefore in P), so it suffices to show that $\Delta^*_{t;T}$ converges to 0 in $L^p(\Omega, \mathcal{F}_t, Q)$. This will also achieve the proof of (30).

Next, for each $T > t$ and $a > 0$ we apply Doob's submartingale inequality (see, e.g., Rogers and Williams (1994, II.70.1)) to the Q-martingale $\Delta_{s;T}$, $s \in [0, t]$. This yields

$$(59) \quad Q\left(\Delta^*_{t;T} > a\right) \leq a^{-1} E^Q |\Delta_{t;T}|,$$

and hence in particular

$$Q\left(\Delta^*_{t;T} > a\right) \leq (a^{-1} E^Q |\Delta_{t;T}|) \wedge 1,$$

where $x \wedge y$ denotes the smaller of x and y. Combining this result with the fact that for any non-negative random variable X and positive p,

$$E^Q X^p = \int_{a=0}^{\infty} p a^{p-1} Q(X > a)\, da,$$

we have that

$$\begin{aligned}
(60) \quad E^Q[|\Delta^*_{t;T}|^p] &= \int_{a=0}^{\infty} p a^{p-1} Q(\Delta^* > a) da \\
&\leq \int_{a=0}^{\infty} p a^{p-1} ((a^{-1} E^Q[|\Delta_{t,T}|]) \wedge 1) da \\
&= \frac{1}{1-p} E^Q[|\Delta_{t;T}|]^p
\end{aligned}$$

In conjunction with (28), this implies that, for any increasing unbounded sequence $\mathcal{T}$, the sequence $\{\Delta^*_{t;T}\}$, $T \in \mathcal{T}$, must converge to 0 in $L^p(\Omega, \mathcal{F}_t, Q)$, and we are done. ∎

Proof of Theorem 2 : Existence of a unique optimum follows from the first part of Theorem 1. Uniform continuity (27) follows from our regularity on u (either (32) or (33)), and existence of all moments of ξ and the martingale change-of-measure follow from the boundedness on compact intervals of r_s, $|\mu_s - r_s|$ and $|\sigma_s^{-1}|$. The rest of the required assumptions are the same.

Proposition 1 (below) shows that we can restrict attention without loss of generality to utility functions u_1 satisfying the smoothness properties (i)–(iv) of Lemma 2 (also below), and we will take them as given from now on.

Independence of returns over time (μ_t, σ_t, and r_t nonstochastic) implies that the conditional distribution of ξ_T/ξ_t conditional on $\mathcal{F}_t$ is the same lognormal distribution as the unconditional distribution. Therefore, dropping the label for the agent for the time being, (53) implies we can write the wealth process as

$$(61)\quad w_{s;T} = h(\xi_s, s, T)$$

where

$$(62)\quad h(x, s, T) \equiv E[\xi_{sT} I(x\lambda_T \xi_{sT})]$$

and $\xi_{sT} \equiv \xi_T/\xi_s$ is the state-price density for purchase at s of a claim at T. The bound (iv) in Lemma 2 on the derivative of I and lognormality of ξ_T/ξ_s allows us to differentiate under the expectation to obtain

$$\begin{aligned}(63)\quad h_x(x, s, T) &= E[\lambda_T \xi_{sT}^2 I'(x\lambda_T\xi_{sT})] \\ &= -x^{-1} E[\xi_{sT}\frac{I(x\lambda_T\xi_{sT})}{R(I(x\lambda_T\xi_{sT}))}].\end{aligned}$$

In the risk-neutral probabilities, the discounted wealth process is a local martingale, and from (56) it is

$$(64)\quad d(\beta_s w_{s;T}) = \beta_s \theta'_{s;T}\sigma_s dZ_s^Q,$$

whereas expanding discounted wealth $\beta_s w_{s;T} = \beta_s h(\xi_s, s, T)$ using Itô's lemma and the various definitions ((16), (12), (13), and $\Sigma \equiv \sigma\sigma'$) yields

$$(65)\quad d(\beta_s w_{s;T}) = -\beta_s h_x(\xi_s, s, T)\xi_s(\mu_s - r_s\mathbf{1})'\Sigma_s^{-1}\sigma_s dZ_s^Q.$$

Matching coefficients, the portfolio process must be essentially

$$(66)\quad \theta_{s;T} = -\xi_s h_x(\xi_s, s, T)\Sigma_s^{-1}(\mu_s - r_s\mathbf{1}).$$

For agent 0, whose relative risk aversion is constant and equal to R, (63) and (61) imply that $-\xi_s h_x(\xi_s, s, T) = R^{-1} w_{s;T}$, and we obtain the standard expression (37) for the reference portfolio process. The form (38) of the reference wealth process also follows easily.

Shortly we will prove (34), but first let us explain how the remaining conclusions of the theorem will then follow.

Since $\beta_t(w_{1t;T} - w_{0t;T})$ is a Q-martingale, Doob's L^2-martingale maximal inequality (Rogers and Williams [1987], Lemma II.31) and (34) imply that

$$(67) \quad E^Q[\sup_{s\le t} \beta_t^2 (w_{1t;T} - w_{0t;T})^2] \stackrel{T\uparrow\infty}{\longrightarrow} 0,$$

and since β_s is nonstochastic and bounded below away from zero on $[0,t]$, (35) follows immediately.

To verify (36), first note from (63) that

$$\xi_s^2(h_x^1(\xi_s, s, T) - h_x^0(\xi_s, s, T)) = -E_s\left[\xi_T\left(\frac{I_1(\lambda_{1T}\xi_T)}{R_1(I_1(\lambda_{1T}\xi_T))} - \frac{I_0(\lambda_{0T}\xi_T)}{R_0(I_0(\lambda_{0T}\xi_T))}\right)\right]$$

is a P-martingale and therefore $\phi_{s;T} \equiv \beta_s\xi_s(h_x^1(\xi_s, s, T) - h_x^0(\xi_s, s, T))$ is a Q-martingale (as will be important because of ϕ's close connection to portfolio choice in (66)), and therefore by Jensen's inequality $E^Q\phi_{s;T}^2$ is nondecreasing in s. This provides the initial inequality in the following, which is also based on (65), (66), the definition of κ_s, and the assumptions that σ is nonsingular and $\mu - r\mathbf{1}$ is nonzero:

$$\begin{aligned}(68) \quad (E^Q\phi_{t;T}^2)\int_{s=t}^{t+1}\kappa_s ds &\le E^Q\int_{s=t}^{t+1}\phi_{s;T}^2\kappa_s ds \\ &< E^Q\int_{s=0}^{t+1}\phi_{s;T}^2\kappa_s ds \\ &= E^Q\int_{s=0}^{t+1}\beta_s^2(\theta_{1s;T} - \theta_{0s;T})'\Sigma_s(\theta_{1s;T} - \theta_{0s;T})ds \\ &= E^Q\beta_{t+1}^2(w_{1,t+1;T} - w_{0,t+1;T})^2.\end{aligned}$$

Given (34) is true for arbitrary t (including $t+1$), the last expression must go to zero as T increases. But the first expression is just $E^Q[\phi_{t;T}^2]$ multiplied by a positive constant. Since $\phi_{s;T}$ is a martingale, Doob's L^2-martingale maximal inequality (cited above) implies that $E^Q \sup_{s\in[0,t]} \phi_{s;T}^2$ also tends to zero as T increases. The result (36) follows from the definition of ϕ, (66), the fact that β_s is nonstochastic and bounded below away from 0 on $[0,t]$, and the fact that $|\mu - r\mathbf{1}|$ and $|\sigma^{-1}|$ are nonstochastic and assumed bounded on $[0,t]$.

The convergence of portfolio proportions (40) follows directly from (35), (36), (37), (38), and the equivalence of P and Q for random variables measurable with respect to $\mathcal{F}_t$.

We have left only to verify (34). We do this by bounding

$$\text{(69)} \quad E^Q\beta_t^2(w_{1t;T} - w_{0t;T})^2 = E^Q \int_{s=0}^{t} \beta_s^2 \kappa_s(\xi_s(h_x^1 - h_x^0)(\xi_s, s, T))^2 ds$$

using the following bound. Fix arbitrary $\varepsilon > 0$ and choose D such that $C > D$ implies $|R_1(C)^{-1} - R^{-1}| < \varepsilon$ (as we can by (iii) of Lemma 2). From two parts of (63), the arguments used to derive them, and the bound (iv) in Lemma 2,[11]

$$\begin{aligned}
\text{(70)} \quad |x(h_x^1 - h_x^0)(x,t,T)| = \Big| & E[x\xi_{tT}^2 \lambda_{1T} I_1'(x\lambda_{1T}\xi_{tT}) + R^{-1}\xi_{tT} I_1(x\lambda_{1T}\xi_{tT}) \\
& : I_1(x\lambda_{1T}\xi_{tT}) \le D] \\
& - E\left[\xi_{tT}\left\{\frac{I_1(x\lambda_{1T}\xi_{tT})}{R_1(I_1(x\lambda_{1T}\xi_{tT}))} - \frac{I_1(x\lambda_{1T}\xi_{tT})}{R}\right\} : I_1(x\lambda_{1T}\xi_{tT}) \ge D\right] \\
& - R^{-1}E[\xi_{tT}(I_1(x\lambda_{1T}\xi_{tT}) - I_0(x\lambda_{0T}\xi_{tT}))]\Big| \\
\le \quad & \left|E[x\xi_{tT}^2\lambda_{1T}(x\lambda_{1T}\xi_{tT})^{-1}A'(1 + (u_1'(D))^{-\gamma'})]\right| + \left|R^{-1}E[\xi_{tT}D]\right| \\
& + |E[\xi_{tT} I_1(x\lambda_{1T}\xi_{tT})\varepsilon]| + R^{-1}\left|h^1(x,t,T) - h^0(x,t,T)\right| \\
= \quad & A'(1 + (u'(D))^{-\gamma'})E[\xi_{tT}] + R^{-1}DE[\xi_{tT}] + \varepsilon h^1(x,t,T) \\
& + R^{-1}|h^1(x,t,T) - h^0(x,t,T)|
\end{aligned}$$

[11] Recall the notation $E[x : y]$ is the same integral as $E[x]$ except with domain limited to the set on which y is true.

$$
\begin{aligned}
&\leq \ A'(1+(u'(D))^{-\gamma'})E[\xi_{tT}] + R^{-1}DE[\xi_{tT}] + \varepsilon h^0(x,t,T) \\
&\quad + (R^{-1}+\varepsilon)|h^1(x,t,T) - h^0(x,t,T)| \\
&\leq \ \varepsilon + \varepsilon h^0(x,t,T) + (R^{-1}+\varepsilon)|h^1(x,t,T) - h^0(x,t,T)|,
\end{aligned}
$$

for T sufficiently large (by (24)). With (69) this implies that

$$
\begin{aligned}
(71)\quad \psi(t) \ &\equiv \ E^Q \beta_t^2 (w_{1t;T} - w_{0t;T})^2 \\
&\leq \ E^Q \int_{s=0}^t \beta_s^2 \kappa_s (\varepsilon + \varepsilon h^0(\xi_s, s, T) \\
&\quad + (R^{-1}+\varepsilon)|h^1(\xi_s,s,T) - h^0(\xi_s,s,T)|)^2 ds \\
&= \ E^Q \int_{s=0}^t \beta_s^2 \kappa_s (\varepsilon + \varepsilon w_{0s;T} + (R^{-1}+\varepsilon)|w_{1s;T} - w_{0s;T}|)^2 ds \\
&\leq \ 4\int_{s=0}^t \kappa_s ((R^{-1}+\varepsilon)^2 \psi(s) + \beta_s^2 \varepsilon^2 + \beta_s^2 \varepsilon^2 E^Q[w_{0s;T}^2]) ds.
\end{aligned}
$$

Gronwall's lemma (Dieudonné [1969], (10.5.1.3)) implies a bound on $\psi(t)$ that can be made arbitrarily small since $\varepsilon > 0$ is arbitrary, given the form (38) of w_{0t} and the uniform bounds on κ_s (inherited from $|\sigma^{-1}|$ and $|\mu - r\mathbf{1}|$) and β_s on $[0,t]$. This completes the proof. ∎

Here is the Lemma that tells us that there is a smoothed version of u_1 with nice properties. There are many ways of performing the smoothing; the particular choice here is one that is useful in Proposition 1 that is used in the proof of Theorem 2.

Lemma 2 *Suppose the von Neumann-Morgenstern utility function u is strictly increasing and strictly concave and that the marginal utility is regularly varying at infinity with index $-R$ for some $R > 0$, that is, $(\forall a > 0) \lim_{C\uparrow\infty} u'(aC)/u'(C) = a^{-R}$. Let I be the inverse of u' (as before). Fix $\alpha > 0$ and define*

$$
(72)\quad \tilde{I}(x) \equiv \int_{-\infty}^{\infty} \exp(\frac{-y^2}{2\alpha}) I(xe^y) \frac{dy}{\sqrt{2\pi\alpha}},
$$

and let $\tilde{u}(C)$ be any integral of the inverse of $\tilde{I}$. Then,

(i) $\tilde{I} \in C^\infty$,

(ii) $\tilde{I}(x)/I(x) \to \exp(\alpha R^2/2)$ *as* $x \downarrow 0$,

(iii) $\tilde{R}(C) \equiv -C\tilde{u}''(C)/\tilde{u}'(C) \to R$ *as* $C \uparrow \infty$, *and*

(iv) $(\exists A' > 0, \gamma' > 0) \quad 0 < \tilde{I}'(x) \leq x^{-1}A'(1 + x^{-\gamma'})$.

Proof The proof builds on basic properties of regularly varying functions given by Bingham, Goldie, and Teugels [1987], henceforth *BGT*. Since u' is decreasing and regularly varying with index $-R$, it follows that I is decreasing and regularly varying with index $-1/R$ at the origin (by the definitions in BGT, section 1.4.2 and the inversion theorem 1.5.12, in which the actual inverse is an asymptotic inverse). Most of the results in BGT are stated for regular variation around infinity; to apply them to I they must be translated through the definitions in BGT, section 1.4.2. Specifically, $I(x)$ regularly varying of index $-1/R$ at the origin is equivalent to $I(1/x)$ regularly varying of index $1/R$ at infinity.

To show the existence of certain integrals, it will be useful to note an implication of Potter's bound (BGT, Theorem 1.5.6.iii) and positivity and monotonicity of I. Then there exists x^* such that for all $y > 0$ and all x, $0 < x \leq x^*$,

$$(73) \quad 0 \leq \frac{I(y)}{I(x)} \leq 2\max\left\{(y/x)^{-1/R-1}, 1\right\},$$

where the first argument of the maximum comes from Potter's bound and the second argument comes from monotonicity of I. In particular, setting $x = x^*$ implies that for all $y > 0$,

$$(74) \quad 0 \leq I(y) < 2I(x^*)((y/x^*)^{-1/R-1} + 1).$$

(i) We can rewrite (72) as

$$(75)\quad \tilde{I}(x) = \int_{-\infty}^{\infty} \exp(-(y-\log(x))^2/2\alpha) I(e^y)dy/\sqrt{2\pi\alpha},$$

and therefore the result follows from (74).

(ii) By (72),

$$(76)\quad \frac{\tilde{I}(x)}{I(x)} = \int_{y=-\infty}^{\infty} e^{-y^2/2\alpha} \frac{I(xe^y)}{I(x)} \frac{dy}{\sqrt{2\pi\alpha}}.$$

By regular variation, $\lim_{x\downarrow 0} I(xe^y)/I(x) = e^{-y/R}$, and therefore the integral of the pointwise limit of the integrand is $e^{\alpha/2R^2}$. Substituting in the bound (73) implies the integrand is integrable uniformly in x, so by Lebesgue's Dominated Convergence Theorem the limit of the integral is the integral of the limit.

(iii) Since $\tilde{R}(c) = -d\log(\tilde{u'}(c))/d\log(c)$ and $\tilde{I}$ is the inverse of $\tilde{u'}$, $\tilde{R}(\tilde{I}(x)) = (-d\log(\tilde{I}(x))/d\log(x))^{-1} = (-x\tilde{I}'(x)/\tilde{I}(x))^{-1}$. Therefore, we want to show that as $x \downarrow 0$ (so $I(x) \uparrow \infty$), R is the limit of

$$(77)\quad \tilde{R}(x) = \left(\frac{-x\tilde{I}'(x)/I(x)}{\tilde{I}(x)/I(x)}\right)^{-1}.$$

We know from (ii) that the limit of the denominator is $e^{\alpha/2R^2}$, so we need to show that the numerator $\mathcal{N}(x)$ tends to $-e^{\alpha/2R^2}/R$ as $x \downarrow 0$. From (75),

$$\begin{aligned}(78)\quad \mathcal{N}(x) &= -\frac{x}{I(x)}\frac{d}{dx}\int_{y=-\infty}^{\infty} \exp(-(y-\log(x))^2/2\alpha)I(e^y)dy/\sqrt{2\pi\alpha} \\ &= -\frac{x}{I(x)}\int_{-\infty}^{\infty} \frac{y-\log(x)}{x\alpha}\exp(-(y-\log(x))^2/2\alpha)I(e^y)dy/\sqrt{2\pi\alpha} \\ &= -\int_{-\infty}^{\infty} \frac{y}{\alpha}\exp(-y^2/2\alpha)\frac{I(xe^y)}{I(x)}dy/\sqrt{2\pi\alpha}.\end{aligned}$$

By regular variation, $\lim_{x\downarrow 0} I(xe^y)/I(x) = e^{-y/R}$, and therefore the integral of the pointwise limit of the integrand is $e^{\alpha/2R^2}/R$. Substituting in the bound (73) implies the integrand is integrable uniformly in x, so by Lebesgue's Dominated Convergence Theorem the limit of the integral is the integral of the limit.

(iv) Since the numerator $\mathcal{N}(x) \equiv -x\tilde{I}'(x)/\tilde{I}(x)$, (78) is equivalent to

$$(79) \quad \tilde{I}'(x) = -\frac{I(x)}{x\alpha}\int_{-\infty}^{\infty} y \exp(-y^2/2\alpha)\frac{I(xe^y)}{I(x)}dy/\sqrt{2\pi\alpha}.$$

The integral can be bounded independently of x by substituting (73) into the integrand, and the result follows from (74). ∎

Proposition 1 *Under the assumptions of Theorem 2 and given fixed t, the wealth processes and optimal demands for all $s \in [0, t]$ and all T sufficiently large are the same as for a different problem satisfying the same assumptions and for which the utility functions satisfy additionally the smoothness properties (i)–(iv) of Lemma 2.*

Proof The intuition of the proof is that the stochastic evolution of ξ implies that the implied preferences at a point in time before the end are smoother than the preferences at the end. To make the smoothing comparable for different T, it is easiest to consider the smoothing over some fixed time interval after t (we consider specifically the interval $[t, t+1]$) rather than an interval ending at T that might not be directly comparable with an interval ending at a different T.

We take as given the (nonstochastic) processes for μ, r, and σ, and will specify new (nonstochastic) processes $\hat{\mu}$, $\hat{r}$, and $\hat{\sigma}$, and a new utility function $\hat{u}_i$ preserving the properties of preferences in the theorem but also satisfying the smoothness properties (i)–(iv) of Lemma 2.

Define $v \equiv \int_{s=t}^{t+1} (\mu_s - r_s \mathbf{1})' \Sigma_s^{-1} (\mu_s - r_s \mathbf{1}) ds$, which is equal to the variance of $\log(\xi_{t+1}/\xi_t)$ in the original problem. In the new problem, we will take $\hat{u}_i$ to be an integral of the inverse of

$$(80) \quad \hat{I}(x) = \int_{z=-\infty}^{\infty} e^z I(xe^z) e^{-z^2/2v} \frac{dz}{\sqrt{2\pi v}}.$$

It is easy to verify (by completing the square in the exponent) that $\hat{I}(x) = e^{v/2} \tilde{I}(xe^v)$, where $\tilde{I}$ is defined in the statement of Lemma 2, and therefore $\hat{I}$ inherits the required smoothness properties (i)–(iv) from $\tilde{I}$.

For the return processes, we want to make the noise in $\hat{\xi}$ plus the noise already embedded into $\hat{I}$ the same as the noise in ξ, so that for all $T > t+2$ and all $s \in [0,t]$, $\log(\xi_T/\xi_s))$ has the same distribution as $z + \log(\hat{\xi}_T/\hat{\xi}_s)$, where $z \sim N(0,v)$ is drawn independently of $\hat{\xi}$. We will also make sure that ξ_s and $\hat{\xi}_s$ are identical for $s \in [0,t]$. All this will ensure that the Lagrange multiplier λ_i as in(49) and consequently the wealth process as in (53) and the implementing portfolio process are the same in the new problem, and we will be done. To accomplish this, we define

$$(81) \quad \hat{r}_s \equiv r_s,$$

$$(82) \quad \hat{\sigma}_s \equiv \begin{cases} \sigma_{(s+t+2)/2} & \text{for } s \in [t, t+2] \\ \sigma_s & \text{otherwise} \end{cases},$$

and

$$(83) \quad \hat{\mu}_s \equiv \begin{cases} r_s + \frac{1}{\sqrt{2}}(\mu_{(s+t+2)/2} - r_{(s+t+2)/2}) & \text{for } s \in [t, t+2] \\ \mu_s & \text{otherwise} \end{cases}.$$

References

Bingham, N. H., C. M. Goldie, and J. L. Teugels, 1987, *Regular Variation*, Cambridge University Press.

Cass, David, and Joseph Stiglitz, 1970, The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation, *Journal of Economic Theory* **2**, 122–160.

Cox, John C., and Chi-fu Huang, 1992, A Continuous-Time Portfolio Turnpike Theorem, *Journal of Economic Dynamics and Control* **16**, 491–507.

Dieudonné, J., 1969, *Foundations of Modern Analysis*, Academic Press.

Dorfman, Robert, Paul A. Samuelson, and Robert M. Solow, 1958, *Linear Programming and Economic Analysis*, McGraw-Hill.

Hakansson, Nils H., 1974, Convergence to Isoelastic Utility and Policy in Multiperiod Portfolio Choice, *Journal of Financial Economics* **1**, 201-224.

Huang, Chi-fu, and Thaleia Zariphopoulou, forthcoming, Turnpike behavior of long-term investments, *Finance and Stochastics.*

Huberman, Gur, and Stephen A. Ross, 1983, Portfolio Turnpike Theorems, Risk Aversion, and Regularly Varying Functions, *Econometrica* **51**, 1345-1361.

Karatzas, I., 1989, Optimization problems in the theory of continuous trading. *SIAM J. Control and Opt.* **27**, 1221-1259.

Leland, Hayne, 1972, On Turnpike Portfolios, in G. Szego and K. Shell, eds., *Mathematical Models in Investment and Finance*, North-Holland, Amsterdam.

Mossin, Jan, 1968, Optimal Multiperiod Portfolio Policies, *Journal of Business* **3**, 373-413.

Rogers, L. C. G., and D. Williams, 1987, *Diffusions, Markov Processes and Martingales, Vol. 2*, Wiley.

Rogers, L. C. G., and D. Williams, 1994, *Diffusions, Markov Processes and Martingales, Vol. 1*, Wiley.